# ENGG. MATH II
# LECTURE NOTES
# BY

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## Derivative

Suppose $y$ is a function of $x$ i.e.

$$y = f(x).$$

derivative of $y$ w.r.t. $x$ is denoted as $\frac{dy}{dx}$ or $f'(x)$.

**Derivative by first principle or ab-initio form:**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(1) Suppose $f(x) = x$ then $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}} = 1$$

(2) Let us take $f(x) = x^2$ then $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$

$f(x+h) = (x+h)^2$ | $f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

where $y = x + h$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$\lim_{y \to x} \frac{y^2 - x^2}{y - x} = 2(x)^{n-1} = 2(x)^{2-1} = 2(x) = 2x$

$$= \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} (2x+h) = 2x + 0 = 2x.$$

③ Suppose $f(x) = \sqrt{x} = x^{1/2}$. $f(x+h) = (x+h)^{1/2}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^{1/2} - x^{1/2}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^{1/2} - x^{1/2}}{(x+h) - (x)}$$

Let $y = x + h$ $y \to x$ as $h \to 0$

$$= \lim_{y \to x} \frac{y^{1/2} - x^{1/2}}{y - x}$$

$$= \frac{1}{2}(x)^{\frac{1}{2}-1} = \frac{1}{2}(x)^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

④ Suppose $f(x) = \frac{1}{x}$ $\quad f(x+h) = \frac{1}{x+h}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - h}{x(x+h) \cdot h} =$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(x)(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2}.$$

(8) $f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$ ✓

$= \lim_{h\to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$ ✓

$= \lim_{h\to 0} \frac{\cancel{2}\cos\left(\frac{2x+h}{2}\right)\sin(h/2)}{\cancel{2}\cdot h/2}$ ✓

$= \cos\left(\frac{2x+0}{2}\right) \cdot \lim_{\frac{h}{2}\to 0} \frac{\sin(h/2)}{(h/2)}$

$= \cos(x) \cdot \lim_{t\to 0} \frac{\sin t}{t}$

$= \cos(x) \cdot 1 = \cos x.$

(9) $f(x) = \cos x \quad f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$

$= \lim_{h\to 0} \frac{-\cancel{2}\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{\cancel{x}+h-\cancel{x}}{2}\right)}{\cancel{2}\cdot h/2}$

$= \lim_{h\to 0} -\sin\left(\frac{2x+h}{2}\right) \boxed{\lim_{h\to 0} \frac{\sin(h/2)}{(h/2)}}$ form (8).

$= -\sin\left(\frac{\cancel{2}x}{\cancel{2}}\right)(1)$

$= -\sin(x)$

$\boxed{\text{Exercise}}$ find the derivative of $\sec x$, $\operatorname{cosec} x$

Ans: $\rightarrow (\sec x)' = \sec x \tan x$, $(\operatorname{cosec} x)' = -\operatorname{cosec}(x)\cot(x)$.

(5) $f(x) = x^n$.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{(x+h) - x}$$

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = n a^{n-1}$$

put $x+h = y$ $y \to x$ as $h \to 0$

$$= \lim_{y\to x} \frac{y^n - x^n}{y-x}$$

$$= n x^{n-1}$$

$\boxed{\text{Now}}$ you can find derivative of $x^3, x^4, x^5$ etc. easily.

(6) $f(x) = a^x$.

$$a^{m+n} = a^m \cdot a^n.$$

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x (a^h) - a^x \cdot 1}{h}$$

$$\boxed{\because \lim_{x\to 0} \frac{a^x - 1}{x} = \log_e^a}$$

$$= \lim_{h\to 0} \frac{a^x (a^h - 1)}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x \log_e^a = a^x \ln a.$$

Next take $f(x) = e^x \Rightarrow f'(x) = e^x \ln e = e^x$ ($\because \ln e = 1$).

(7) $f(x) = \log_e^x = \ln x$.

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0} \ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

$$= \ln\left[\lim_{h\to 0} \left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]^{\frac{1}{x}} = \ln e^{1/x} = \frac{1}{x} \ln e = \frac{1}{x}.$$

If $f(x) = \tan x$ then $f'(x) = \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$ ①

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{\cos x \cdot \cos(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{\sin(\cancel{x}+h-\cancel{x})}{h \cos x \cdot \cos(x+h)}$$

$$= \lim_{h\to 0}\left(\frac{\sin h}{h}\right) \cdot \lim_{h\to 0} \frac{1}{\cos x \cdot \cos(x+h)}$$

$$= (1) \cdot \frac{1}{\cos x \cdot \cos x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Ex:→ find the derivative of $\cot x$. A→ $-\operatorname{cosec}^2 x$.

**Next** (i) $\frac{d}{dx}(\text{any constant}) = 0$.

(ii) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$

(iii) $\frac{d}{dx}(f(x)\, g(x)) = f(x)\frac{d}{dx}(g(x)) + \frac{d}{dx}(f(x)) \cdot g(x)$.

(iv) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x) \cdot \frac{d}{dx}(g(x))}{\{g(x)\}^2}$

(iv) result is true for only those $x : g(x) \neq 0$.

(v) $\frac{d}{dx}(af(x) \pm bg(x)) = a\frac{d}{dx}(f(x)) \pm b \cdot \frac{d}{dx}(g(x))$.

Let us talk about some simple exercises.

(i) $\frac{d}{dx}(x^7 + \log_e^x + e^x + a^x)$. (ii) $\frac{d}{dx}(\sin^2 x)$

③ $\frac{d}{dx}\left(\frac{1-\tan x}{1+\tan x}\right) = ?$ ④ $\frac{d}{dx}(\sin x + \cos x)^2 = ?$

⑤ $\frac{d}{dx}(\sin x - \cos x)^2$ ⑦ $(2x+1)^2$ ⑧ $x^7 - 10x^{11} + 5\tan x$

Next step is to discuss chain rule

i.e. $y = \phi(t) \quad t = \psi(x)$.

then $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$.

ⓐ Let us take a simple example $y = (x^2+2)^3$.

put $t = x^2+2$ then $y = t^3$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{d}{dt}(t^3) \cdot \frac{d}{dx}(x^2+2)$$

$$= 3t^2 (2x) = 3(2x)(x^2+2)^2.$$

ⓑ Suppose $y = \sin^2 x$.

put $t = \sin x \Rightarrow y = t^2$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{d}{dt}(t^2) \cdot \frac{d}{dx}(\sin x)$$

$$= 2t \cos x$$

$$= 2\sin x \cos x.$$

$\boxed{ex}$ Differentiate $\cos^2 x$, $\tan^2 x$, $\cot^2 x$, $\sec^2 x$, $\operatorname{cosec}^2 x$.

ⓒ $y = \log(\sec x)$ put $\sec x = t \Rightarrow y = \ln(t)$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{d}{dt}\log(t) \cdot \frac{d}{dx}(\sec x).$$

$$= \frac{1}{t} \cdot (\sec x \tan x)$$

$$= \frac{\sec x \tan x}{\sec x} = \tan x.$$

(ii)

(d) $y = \sqrt{\sin x}$.

put $\sin x = t$ rest is easy.

Ans: $\rightarrow$ $\dfrac{\cos x}{2\sqrt{\sin x}}$

(e) $y = e^{\sin x} \Rightarrow y = e^{t}$

put $\sin x = t \Rightarrow \dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$

$$= \frac{d}{dt}(e^{t}) \cdot \frac{d}{dx}(\sin x)$$

$$= e^{t} \cos x$$

$$= e^{\sin x} \cos x.$$

(f) $y = \sqrt{\sin(\sqrt{x})}$.

put $\sqrt{x} = t \Rightarrow \underline{y = \sqrt{\sin t}}$

put $\sin t = v. \Rightarrow y = \sqrt{v}$.

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{d}{dv}(\sqrt{v}) \cdot \frac{d}{dt}(\sin t) \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{v}} \cdot \cos(t) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\cos(\sqrt{x})}{4\sqrt{x}\sqrt{\sin x}}.$$

# Differentiate the following

(1)

~~Answer~~

(1) $2\sqrt{x} - 3e^{x} + 7\sin x - 8\cos x + 11$

(2) $\sqrt{x}\log x$ (3) $x^2\operatorname{cosec} x$ (4) $(x^3-4)\tan x$

(5) $\dfrac{x-7}{x-3}$ (6) $\dfrac{e^x}{\sin x}$ (7) $\dfrac{\tan x}{x^4}$ (8) $\dfrac{1-\cos x}{1+\sin x}$

(9) $\dfrac{4}{x} + 7\cos x - 9\log x + 8\sin x - 3$

(10) $\sqrt{x}\log x\operatorname{cosec} x$ (11) $x^3\cot x\log x$

(12) $(2x+5)^3$ (13) $\sqrt{ax+b}$ (14) $\sin 2x$

(15) $\cos^3 x$ (16) $\sin^2(5x)$ (17) $\sin(\cos x)$

(18) $\cos(\log x)$ (19) $\log(2x+3)$ (20) $e^{x^2}$

## Derivative of Inverse trigonometric functions



① $y = \sin^{-1}x \quad x \in [-1, 1] \quad y \in [-\pi/2, \pi/2]$.

$\Rightarrow x = \sin y$

$\Rightarrow \frac{dx}{dy} = \cos y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$.

② $y = \cos^{-1}x \quad x \in [-1, 1] \quad y \in [0, \pi]$.

$\Rightarrow x = \cos y$

$\Rightarrow \frac{dx}{dy} = -\sin y$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$.

③ $y = \tan^{-1}x \quad x \in R \quad y \in (-\frac{\pi}{2}, \pi/2)$.

$\Rightarrow x = \tan y$

$\Rightarrow \frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$.

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$

④ $y = \cot^{-1}x \quad x \in R \quad y \in (0, \pi)$.

$\Rightarrow x = \cot y$

$\Rightarrow \frac{dx}{dy} = -\operatorname{cosec}^2 y = -(1 + \cot^2 y) = -(1 + x^2)$.

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-1}{1+x^2}}$

(5) $y = \sec^{-1}x \cdot \quad x \in (-\infty, -1] \cup [1, \infty) \cdot \quad \cancel{y \in [\frac{-\pi}{2}, 0) \cup (0, \pi/2]}$

$y \in [0, \pi/2) \cup (\pi/2, \pi] \cdot$

$\Rightarrow x = \sec y$

$\Rightarrow \frac{dx}{dy} = \sec y \tan y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y \sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2-1}}$

($\because$ Secy tany is positive in the required intervals)

$$\boxed{\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}}$$

(6) $y = \operatorname{Cosec}^{-1}x \quad x \in (-\infty, -1] \cup [1, \infty) \quad y \in [\frac{-\pi}{2}, 0) \cup (0, \pi/2]$

$\Rightarrow x = \operatorname{Cosec} y$

$\Rightarrow \frac{dx}{dy} = -\operatorname{Cosec} y \cot y$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{Cosec}(y)\cot(y)} = \frac{-1}{\operatorname{Cosec}(y)\sqrt{\operatorname{Cosec}^2(y)-1}} = \frac{-1}{x\sqrt{x^2-1}}$

$\because$ Cosec y Coty is positive in the required intervals

$$\boxed{\frac{dy}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}}$$

$\boxed{\text{V.V Imp}}$ (i)

ex: $\rightarrow$ $\dfrac{d}{dx}\left[\dfrac{1-\tan x}{1+\tan x}\right]^{1/2} = \dfrac{-1}{\sqrt{\cos 2x}\,(\cos x+\sin x)}$

Sol: $\rightarrow$ $y = \left[\dfrac{1-\tan x}{1+\tan x}\right]^{1/2}$

put $t = \dfrac{1-\tan x}{1+\tan x} \Rightarrow y = \sqrt{t}$.

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{d}{dt}(\sqrt{t})\cdot\frac{d}{dx}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$= \frac{1}{2\sqrt{t}}\left[\frac{(1+\tan x)\frac{d}{dx}(1-\tan x)-(1-\tan x)\cdot\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}\right]$$

$$= \frac{1}{2\sqrt{t}}\left[\frac{(1+\tan x)(-\sec^2 x)-(1-\tan x)(\sec^2 x)}{(1+\tan x)^2}\right]$$

$$= \frac{1}{2\sqrt{t}}\left[\frac{-\sec^2 x\,(1+\cancel{\tan x}+1-\cancel{\tan x})}{(1+\tan x)^2}\right]$$

$$= \frac{-\cancel{2}\sec^2 x}{\cancel{2}\sqrt{t}\,(1+\tan x)^2}$$

$$= \frac{-(1+\tan^2 x)}{\sqrt{\frac{1-\tan x}{1+\tan x}}\,(1+\tan x)^{1/2}(1+\tan x)^{3/2}}$$

$$= \frac{-(1+\tan^2 x)}{\sqrt{1-\tan x}\,\sqrt{1+\tan x}\,(1+\tan x)}$$

$$= \frac{-(1+\tan^2 x)}{\sqrt{1-\tan^2 x}\,(1+\tan x)}$$

$$= \frac{-(1+\tan^2 x)\sqrt{1-\tan^2 x}}{(1-\tan^2 x)(1+\tan x)}$$

$$= \frac{-\sqrt{1-\tan^2 x}}{\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)(1+\tan x)}$$

$$= \frac{-\sqrt{1-\tan^2 x}}{(\cos 2x)\left(1+\frac{\sin x}{\cos x}\right)}$$

$$= \frac{-\sqrt{1-\frac{\sin^2 x}{\cos^2 x}}}{\cos 2x\left(\frac{\cos x+\sin x}{\cos x}\right)}$$

$$= \frac{-\sqrt{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}}}{\cos 2x\left(\frac{\cos x+\sin x}{\cos x}\right)}$$

$$= \frac{-\frac{\sqrt{\cos 2x}}{\cancel{\cos x}}}{\cos 2x\left(\frac{\cos x+\sin x}{\cancel{\cos x}}\right)}$$

$$= \frac{-1}{\sqrt{\cos 2x}\,(\cos x+\sin x)}$$

(2) $\frac{d}{dx}\left(\ln \tan(\pi/4 + x/2)\right) = ?$

$y = \ln\left(\tan(\pi/4 + x/2)\right)$

put $\frac{\pi}{4} + \frac{x}{2} = v$, $\tan(\pi/4 + x/2) = t$

$\therefore\ y = \ln t$ where $t = \tan(v)$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dv} \cdot \frac{dv}{dx}$$

$$= \frac{d}{dt}(\ln t) \cdot \frac{d}{dv} \tan\boxed{\overset{v}{(\pi/4 + x/2)}} \cdot \frac{d}{dx}(\pi/4 + x/2)$$

$$= \frac{1}{t} \cdot \sec^2(\cancel{\pi/4}\ v) \cdot \frac{1}{2}$$

$$= \frac{1 + \tan^2 v}{2t} = \frac{1 + \tan^2(\pi/4 + x/2)}{2 \tan(\pi/4 + x/2)}$$

if $\pi/4 + x/2 = \theta \Rightarrow \frac{dy}{dx} = \frac{1 + \tan^2\theta}{2\tan\theta} = \operatorname{Cosec} 2\theta$

$\therefore\ \frac{dy}{dx} = \operatorname{Cosec} 2(\pi/4 + x/2) = \operatorname{Cosec}(\pi/2 + x) = \sec x$

(3) $\frac{d}{dx}(\sin 5x \cos 7x) = \sin 5x \frac{d}{dx}(\cos 7x) + \frac{d}{dx}(\sin 5x) \cos(7x)$

$= \sin 5x(-7\sin 7x) + 5\cos(5x)\cos(7x)$

(4) $\frac{d}{dx}(\sin^{-1}2x) = ?$

Sol$^n$:→ $y = \sin^{-1} 2x.$

put $2x = v \Rightarrow y = \sin^{-1} v.$

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx} = \frac{d}{dv}(\sin^{-1} v)\cdot\frac{d}{dx}(2x)$$

$$= \frac{1}{\sqrt{1-v^2}}(2) = \frac{2}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}.$$

(5) $\frac{d}{dx}\left(\tan^{-1}(\sin^2 x)\right)$

Sol$^n$:→ $y = \tan^{-1}(\sin^2 x).$

put $\sin^2 x = v \Rightarrow \cancel{\sin^2 x = v}\ y = \tan^{-1}(v).$

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx} = \frac{d}{dv}(\tan^{-1} v)\cdot\frac{d}{dx}(v).$$

$$= \frac{1}{1+v^2}\cdot\frac{d}{dx}(\sin^2 x)$$

$$= \frac{1}{1+(\sin^2 x)^2}\left[2\sin x\cos x\right]$$

$$= \frac{2\sin x\cos x}{1+\sin^4 x}.$$

(6) Diff. $\tan^{-1}\left(\frac{1}{1-x+x^2}\right)$

Sol$^n$:→ $\tan^{-1}\left(\frac{1}{1-x+x^2}\right) = \tan^{-1}\left(\frac{1}{1+x^2-x}\right)$

$$= \tan^{-1}\left(\frac{1}{1+x(x-1)}\right) = \tan^{-1}\left(\frac{x-(x-1)}{1+x(x-1)}\right)$$

$= \tan^{-1}(x) - \tan^{-1}(x-1)$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x) - \frac{d}{dx}\tan^{-1}(x-1)$$

$$= \frac{1}{1+x^2} - \frac{1}{1+(x-1)^2} \cdot \frac{d}{dx}(x-1).$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2-2x+1}$$

$$= \frac{1}{x^2+1} - \frac{1}{x^2-2x+2}$$

Ex: $\rightarrow$ $x^2 \operatorname{cosec}^{-1}\left(\frac{1}{\ln x}\right)$, $(x\sin^{-1}x)^{15}$, $\tan^{-1}(\cos\sqrt{x})$

## Differentiation by Substitution

Sometimes with proper Substitution we can transform the given function to a simpler function in the new variable so the process of differentiation w.r.t. the new variable becomes easier. Finally we apply chain rule to obtain the derivative w.r.t. the original variable.

$\boxed{e.g}$ $\cos^{-1}(4x^3-3x)$ put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$$\therefore y = \cos^{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}\cos 3\theta = 3\theta$$

$$= 3\cos^{-1}x \quad \therefore \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}.$$

1) $\sin^{-1} 2x\sqrt{1-x^2}$

put $x = \sin\theta$

$\therefore y = \sin^{-1} 2\sin\theta\sqrt{1-\sin^2\theta}$

$= \sin^{-1} 2\sin\theta\cos\theta$

$= \sin^{-1}\sin 2\theta$

$= 2\theta = 2\sin^{-1}x.$

$\therefore \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

(2) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow y = 2\tan^{-1}x \Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}.$

(3) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ put $x = \tan\theta$

A: $\rightarrow \frac{2}{1+x^2}$

(4) $\cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$ put $t = \tan\theta$

A: $\rightarrow \frac{2}{1+x^2}.$

Q:→ $\cos^{-1}(2t^2-1)$

put $t = \cos\theta \Rightarrow \theta = \cos^{-1} t$.

$$y = \cos^{-1}(2\cos^2\theta - 1) = \cos^{-1}(\cos 2\theta) = 2\theta = 2\cos^{-1} t.$$

$$\Rightarrow \frac{dy}{dt} = \frac{-2}{\sqrt{1-t^2}}$$

Q:→ $\left[\left(\frac{1+t^2}{1-t^2}\right)^2 - 1\right]^{1/2}$

$$y = \left[\frac{(1+t^2)^2 - (1-t^2)^2}{(1-t^2)^2}\right]^{1/2}$$

$$= \left[\frac{4t^2}{(1-t^2)^2}\right]^{1/2}$$

$$= \frac{2t}{1-t^2}$$

$$\frac{dy}{dt} = 2\left[\frac{(1-t^2) - t(-2t)}{(1-t^2)^2}\right]$$

$$= 2\left[\frac{1-t^2+2t^2}{(1-t^2)^2}\right] = \frac{2(1+t^2)}{(1-t^2)^2}.$$

Diff using logarithm

Suppose $y = [f(x)]^{g(x)}$

$\Rightarrow \log y = g(x) \log [f(x)]$.

Q:→ ① $y = x^{x}$

$\Rightarrow \log y = x \log x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}(\log x) + \frac{d}{dx}(x) \log x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \cancel{x} \cdot \frac{1}{\cancel{x}} + \log x$

$\Rightarrow \frac{dy}{dx} = y [1 + \log x] = x^{x}(1 + \log x)$.

② $y = x^{\sin x}$

$\Rightarrow \log y = \sin x \log x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \frac{d}{dx}(\sin x) \log x$

$\Rightarrow \frac{dy}{dx} = y \left[ \frac{\sin x}{x} + \cos x \log x \right]$

$= x^{\sin x} \left[ \frac{\sin x}{x} + \cos x \log x \right]$.

(3) $y = 2^{2^x}$

$\Rightarrow \log y = 2^x \log 2$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = (\log 2)\left(2^x \log 2\right)$

$\Rightarrow \frac{dy}{dx} = (\log 2)\left(2^x \log 2\right)\left(2^{2^x}\right).$

(4) $y = \left(1+\frac{1}{x}\right)^x$

$\Rightarrow \log y = x \log\left(1+\frac{1}{x}\right)$

$\Rightarrow \log y = x \log\left(\frac{1+x}{x}\right)$

$\Rightarrow \log y = x\left[\log(1+x) - \log x\right].$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = x\left[\frac{1}{1+x} - \frac{1}{x}\right] + \log(1+x) - \log x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \cancel{x}\left[\frac{x-(1+x)}{\cancel{x}(1+x)}\right] + \log\left(\frac{1+x}{x}\right)$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{-1}{1+x} + \log\left(\frac{1+x}{x}\right).$

$\Rightarrow \frac{dy}{dx} = y\left[\frac{-1}{1+x} + \log\left(\frac{1+x}{x}\right)\right].$

(5) $y = x^{1/x} + (\sin x)^x$

$y = y_1 + y_2$

$\Rightarrow \frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$.

$y_1 = x^{1/x}$

$\Rightarrow \log y_1 = \frac{1}{x} \log x$

$\Rightarrow \frac{1}{y_1} \cdot \frac{dy_1}{dx} = \log x \left(\frac{-1}{x^2}\right) + \frac{1}{x} \cdot \frac{1}{x}$

$\Rightarrow \frac{1}{y_1} \cdot \frac{dy_1}{dx} = \frac{1}{x^2} (1 - \log x)$.

$\Rightarrow \frac{dy_1}{dx} = y_1 \left[\frac{1-\log x}{x^2}\right] = x^{1/x} \left[\frac{1-\log x}{x^2}\right]$.

$y_2 = (\sin x)^x$

$\Rightarrow \log y_2 = x \log(\sin x)$.

$\Rightarrow \frac{1}{y_2} \cdot \frac{dy_2}{dx} = x \cdot \frac{d}{dx}(\log \sin x) + \frac{d}{dx}(x) \cdot \log(\sin x)$

$\Rightarrow \frac{1}{y_2} \cdot \frac{dy_2}{dx} = x \cdot \frac{\cos x}{\sin x} + \log(\sin x)$

$\Rightarrow \frac{dy_2}{dx} = y_2 \left[x \cdot \cot x + \log(\sin x)\right]$.

$= (\sin x)^x \left[x \cdot \cot x + \log(\sin x)\right]$.

exercise Set

(1) $\dfrac{e^{3x^2}}{\ln \sin x}$ (2) $e^{\sin x} - a^{\cos x}$

(3) $a^{x^2} b^{x^3}$ (4) $5^{\sin x^2}$ (5) $\sqrt{\sec(2x+1)}$

(6) $\sin^2 x \cos^2 x$ (7) $\tan^3 x$ (8) $\sin^4 x$

(9) $\sqrt{a^{\sqrt{x}}}$ (10) $e^{\sqrt{ax}}$ (11) $\cos^{-1}\sqrt{\dfrac{1+x}{2}}$

(12) $\cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$ (13) $\sin^{-1}\left(\dfrac{2\sqrt{t^2-1}}{t}\right)$

(14) $\tan^{-1}\sqrt{\dfrac{1-t}{1+t}}$ (15) $\sec^{-1}\left(\dfrac{\sqrt{a^2+x^2}}{a}\right)$

(16) $x^{\sin x} + \sin x^{x}$ (17) $x^{\cos x} + \cos x^{x}$

(18) $(\sec x + \tan x)^{\cot x}$ (19) $\dfrac{(x+1)(x+2)^2(x+3)^3}{(x-1)(x-2)^2(x-3)^3}$

(20) $\sin^n(x^n)$. (21) $(\sin x)^x \sqrt{\sin x}\,(1+x^2)^{\frac{1}{2}+x}$.

## Parametric differentiation

$x$ and $y$ both are functions of a common parameter $t$ or $\theta$ or $\phi$ etc.

i.e. $x = \phi(t)$ $\quad y = \psi(t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Q: → $x = at^2$, $y = 2at$ $\quad \frac{dy}{dx} = ?$

Sol: → $x = at^2 \Rightarrow \frac{dx}{dt} = 2at.$

$y = 2at \Rightarrow \frac{dy}{dt} = 2a$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}.$

Q: → $x = a\cos^3\theta$ $\quad y = b\sin^3\theta$

Sol: → $x = a\cos^3\theta \Rightarrow \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta).$

$y = b\sin^3\theta \Rightarrow \frac{dy}{d\theta} = 3b\sin^2\theta(\cos\theta).$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cancel{3}b\sin^{\cancel{2}}\theta\cancel{\cos\theta}}{-\cancel{3}a\cos^{\cancel{2}}\theta\cancel{\sin\theta}}$$

$$= \frac{-b}{a}\tan(\theta).$$

Q: → $x = t + \sqrt{t}$ $\quad y = t - \sqrt{t}$

$\frac{dx}{dt} = 1 + \frac{1}{2\sqrt{t}}$ $\quad \frac{dy}{dt} = 1 - \frac{1}{2\sqrt{t}} = \frac{2\sqrt{t}-1}{2\sqrt{t}}$

$= \frac{2\sqrt{t}+1}{2\sqrt{t}}$ $\quad \therefore \frac{dy}{dx} = \frac{2\sqrt{t}-1}{2\sqrt{t}} \cdot \frac{2\sqrt{t}}{2\sqrt{t}+1} = \frac{2\sqrt{t}-1}{2\sqrt{t}+1}$

(4) $x = a(\theta + \sin\theta) \quad y = a(1+\cos\theta)$

$\frac{dx}{d\theta} = a(1+\cos\theta) \quad \frac{dy}{d\theta} = -a\sin\theta$

$\therefore \frac{dy}{dx} = \frac{-\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)} = \frac{-\sin\theta}{1+\cos\theta}$.

$\boxed{\text{exercises}}$ (1) $x = a\cos\theta, \; y = a\sin\theta$

(2) $x = a\cos^3 t, \; y = a\sin^3 t$ at $t = \pi/4$.

Sol$^n$:→ $x = a\cos^3 t \Rightarrow \frac{dx}{dt} = 3a\cos^2 t(-\sin t)$.

$y = a\sin^3 t \Rightarrow \frac{dy}{dt} = 3a\sin^2 t(\cos t)$.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cancel{3}\cancel{a}\sin^{\cancel{2}} t \cancel{\cos t}}{\cancel{3}\cancel{a}\cos^{\cancel{2}} t(-\cancel{\sin t})} = \tan(t)$.

$\left.\frac{dy}{dx}\right]_{t=\pi/4} = \tan(\pi/4) = 1$.

~~(3) $x = 3\cot t - 2\cot^3 t$~~

(3) $x = 3\cos t - 2\cos^3 t, \quad y = 3\sin t - 2\sin^3 t$

find $\frac{dy}{dx} = ?$

Diff. w.r.t. a function

$y = f(x) \quad z = g(x) \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$.

(Q) $\sqrt{x}$ w.r.t $x^2$

$y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \qquad \frac{dy}{dz} = \frac{\frac{1}{2\sqrt{x}}}{2x} = \frac{1}{4x^{3/2}}$.

$z = x^2 \Rightarrow \frac{dz}{dx} = 2x$

(2) $y = \tan^{-1}x, \quad z = \cos^{-1}x$

$$\frac{dy}{dx} = \frac{1}{1+x^2}, \quad \frac{dz}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dz} = \frac{dy}{dx} \Big/ \frac{dz}{dx} = \frac{\frac{1}{1+x^2}}{\frac{-1}{\sqrt{1-x^2}}} = -\frac{\sqrt{1-x^2}}{1+x^2}.$$

(3) $\sin x$ w.r.t. $\cot x$.

Solⁿ:→ $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x.$

$z = \cot x \Rightarrow \frac{dz}{dx} = -\operatorname{cosec}^2 x.$

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\cos x}{-\operatorname{cosec}^2 x} = -\sin^2 x \cos x.$$

(4) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Solⁿ:→ $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$

$= 2\theta = 2\tan^{-1}x.$

$z = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$

$= 2\theta = 2\tan^{-1}x$

$\therefore y = z \Rightarrow \frac{dy}{dz} = 1.$

(5) $\tan^{-1}x$ w.r.t. $\tan^{-1}\sqrt{1+x^2}.$

# Differentiation of Implicit function

$F(x,y)=0$ where $x$ and $y$ both are independent and dependent variables may determine one or more two functions. Any such function is Known as implicit function.

$\boxed{e.g}$ ① $x^2+y^2-a^2=0$ find $\frac{dy}{dx}=?$

Sol$^n$:→ $2x+2y\frac{dy}{dx}-0=0 \Rightarrow 2y\frac{dy}{dx}=-2x$

$$\Rightarrow \frac{dy}{dx}=\frac{-2x}{2y}=\frac{-x}{y}.$$

② $y^3+3x^2y-2x=0$

$$\Rightarrow 3y^2\frac{dy}{dx}+3\left(x^2\frac{dy}{dx}+y(2x)\right)-2=0$$

$$\Rightarrow (3y^2+3x^2)\frac{dy}{dx}=\cancel{2+6xy}\ 2-6xy$$

$$\Rightarrow \frac{dy}{dx}=\frac{2(1-3xy)}{3(x^2+y^2)}.$$

③ $xy^2+x^2y+1=0$

$$\Rightarrow x\left(2y\cdot\frac{dy}{dx}\right)+y^2+x^2\frac{dy}{dx}+y(2x)=0$$

$$\Rightarrow (2xy+x^2)\frac{dy}{dx}=-(2xy+y^2)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y(2x+y)}{x(2y+x)}.$$

④ $x^y = y^x$ find $\frac{dy}{dx}$.

$\Rightarrow y \ln x = x \ln y$

$\Rightarrow \frac{y}{x} + \ln x \frac{dy}{dx} = \frac{x}{y} \cdot \frac{dy}{dx} + \ln y$

$\Rightarrow \left(\ln x - \frac{x}{y}\right)\frac{dy}{dx} = \ln y - y/x$

$\Rightarrow \left(\frac{y \ln x - x}{y}\right)\frac{dy}{dx} = \frac{x \ln y - y}{x}$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{y(x \ln y - y)}{x(y \ln x - x)}}$$

⑤ $x^y y^x = 1$ find $\frac{dy}{dx}$.

$\Rightarrow \ln(x^y y^x) = \ln(1) = 0$

$\Rightarrow \ln(x^y) + \ln(y^x) = 0$

$\Rightarrow y \ln x + x \ln y = 0$

$\Rightarrow \ln x \cdot \frac{dy}{dx} + \frac{y}{x} + \frac{x}{y} \cdot \frac{dy}{dx} + \ln y = 0$

$\Rightarrow \left(\ln x + \frac{x}{y}\right)\frac{dy}{dx} = -\left(\ln y + \frac{y}{x}\right) = -\left(\frac{x \ln y + y}{x}\right)$

$\Rightarrow \left(\frac{y \ln x + x}{y}\right)\frac{dy}{dx} = \frac{-(x \ln y + y)}{x} \Rightarrow \frac{dy}{dx} = \frac{-y}{x}\left(\frac{x \ln y + y}{y \ln x + x}\right).$

$\boxed{\text{exercise}}$ ① $e^{xy} + y\sin x = 1$

② $x = y\ln(xy)$.

③ $y^2\cot x = x^2\cot y$

④ $\ln\sqrt{x^2+y^2} = \tan^{-1}\frac{y}{x}$.

⑤ $e^{xy} + y\sin x = 1$

$\boxed{\text{e.g}}$ $\sin(x+y) = y\cos(x+y)$ prove $\frac{dy}{dx} = \frac{-(1+y^2)}{y^2}$.

$\boxed{\text{Sol}^n}$ $\sin(x+y) = y\cos(x+y)$

$\Rightarrow y = \tan(x+y)$

$\Rightarrow \frac{dy}{dx} = \sec^2(x+y)\left\{1+\frac{dy}{dx}\right\}$

$\Rightarrow \frac{dy}{dx} = \left\{1+\tan^2(x+y)\right\}\left\{1+\frac{dy}{dx}\right\}$.

$\Rightarrow \frac{dy}{dx} = (1+y^2)\left(1+\frac{dy}{dx}\right)$

$\Rightarrow \left\{1-(1+y^2)\right\}\frac{dy}{dx} = 1+y^2$

$\Rightarrow -y^2\frac{dy}{dx} = 1+y^2$

$\Rightarrow \frac{dy}{dx} = \frac{-(1+y^2)}{y^2}$.

Q:→ $\sqrt{1-x^4}+\sqrt{1-y^4}=K(x^2-y^2)$ prove $\frac{dy}{dx}=\frac{x\sqrt{1-y^4}}{y\sqrt{1-x^4}}$.

Sol:→ put $x^2=\cos\theta,\ y^2=\cos\phi$

$\therefore\ \sqrt{1-\cos^2\theta}+\sqrt{1-\cos^2\phi}=K(\cos\theta-\cos\phi)$

$\Rightarrow \sin\theta+\sin\phi=K(\cos\theta-\cos\phi)$

$\Rightarrow \cancel{2}\,\cancel{\sin\left(\frac{\theta+\phi}{2}\right)}\cos\left(\frac{\theta-\phi}{2}\right)=-K\left(\cancel{2}\,\cancel{\sin\frac{\theta+\phi}{2}}\sin\frac{\theta-\phi}{2}\right)$

$\Rightarrow \tan\left(\frac{\theta-\phi}{2}\right)=\frac{-1}{K}$

$\Rightarrow \tan\left(\frac{\phi-\theta}{2}\right)=\frac{1}{k}$

$\Rightarrow \frac{1}{2}(\phi-\theta)=\tan^{-1}\left(\frac{1}{k}\right)$

$\Rightarrow \phi-\theta=2\tan^{-1}\left(\frac{1}{k}\right)$

$\Rightarrow \cos^{-1}(y^2)-\cos^{-1}(x^2)=2\tan^{-1}\left(\frac{1}{k}\right)$.

Diff. both sides w.r.t. x

$\Rightarrow \frac{-1}{\sqrt{1-y^4}}(2y)\frac{dy}{dx}+\frac{2x}{\sqrt{1-x^4}}=0$

$\Rightarrow \frac{x}{\sqrt{1-x^4}}=\frac{y}{\sqrt{1-y^4}}\frac{dy}{dx}$

$\Rightarrow \boxed{\frac{dy}{dx}=\frac{x\sqrt{1-y^4}}{y\sqrt{1-x^4}}}$

# Successive Differentiation

$y = f(x)$

$$\frac{dy}{dx} = f'(x).$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \text{ and so on.}$$

we can write $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$ and so on.

$\boxed{\text{eg}}$ ① $y = x^5 + 4x^3 - 2x^2 + 1$. $y_2 = ?$

Sol$^n$:→ $y_1 = 5x^4 + 12x^2 - 4x$

$y_2 = 20x^3 + 24x - 4.$

② $x = \sin t$, $y = \sin(pt)$ prove that

$(1-x^2)y_2 - xy_1 + p^2y = 0$

Sol$^n$:→ $x = \sin t \Rightarrow t = \sin^{-1}x$

$y = \sin(p\sin^{-1}x)$

$\Rightarrow y_1 = \cos(p\sin^{-1}x) \cdot \frac{p}{\sqrt{1-x^2}}$

$\Rightarrow y_1\sqrt{1-x^2} = p\cos(p\sin^{-1}x)$

$\Rightarrow y_1^2(1-x^2) = p^2\cos^2(p\sin^{-1}x)$

$\Rightarrow y_1^2(1-x^2) = p^2(1-y^2)$

$\Rightarrow$

Diff. both sides w.r.t. x

$y_1^2(-2x) + (1-x^2)\,2y_1y_2 = -p^2(2yy_1)$

$\Rightarrow (1-x^2)y_2 - xy_1 = -p^2y$

$\Rightarrow \boxed{(1-x^2)y_2 - xy_1 + p^2y = 0}$

(3) $y = \tan^{-1}x$ prove that $(1+x^2)y_2 + 2xy_1 = 0$

Sol$^n$: $\rightarrow$ $y = \tan^{-1}x$

$\Rightarrow y_1 = \dfrac{1}{1+x^2}$

$\Rightarrow (1+x^2)y_1 = 1$

Diff. both sides w.r.t. x

$\Rightarrow (1+x^2)y_2 + 2xy_1 = 0$

(4) $2y = x\left(1+\dfrac{dy}{dx}\right)$ prove that $y_2$ is constant.

Sol$^n$: $\rightarrow$ $2y = x(1+y_1)$

$\Rightarrow 2y_1 = 1 + xy_2 + y_1$

$\Rightarrow y_1 = 1 + xy_2$

$\Rightarrow y_2 = xy_3 + y_2$

$\Rightarrow xy_3 = 0$

$\Rightarrow y_3 = 0$

$y_3 = \dfrac{d}{dx}(y_2)$ $\therefore$ $\boxed{y_2 = 0}$

(5) $y = ax\sin x$ prove that $x^2 y_2 - 2x y_1 + (x^2+2)y = 0$

Sol$^n$: $\Rightarrow$ $y = ax\sin x$

$\Rightarrow y_1 = a[x\cos x + \sin x]$ —— (1)

$\Rightarrow y_2 = a[x(-\sin x) + \cos x + \cos x]$.

$\Rightarrow y_2 = -ax\sin x + 2a\cos x$ —— (2)

LHS $x^2 y_2 - 2x y_1 + (x^2+2)y$

$= x^2(-ax\sin x + 2a\cos x) - 2x(ax\cos x + a\sin x) + (x^2+2)(ax\sin x)$

$= -ax^3\cancel{\sin x} + 2ax^2\cancel{\cos x} - 2ax^2\cancel{\cos x} - 2ax\cancel{\sin x} + ax^3\cancel{\sin x} + 2ax\cancel{\sin x}$

$= 0$ (RHS) proved.

(6) $y = e^{m\cos^{-1}x}$ prove that $(1-x^2)y_2 - xy_1 = m^2 y$.

Sol$^n$: $\Rightarrow y = e^{m\cos^{-1}x}$

$\Rightarrow y_1 = e^{m\cos^{-1}x}\dfrac{-m}{\sqrt{1-x^2}}$

$\Rightarrow y_1\sqrt{1-x^2} = -my$

$\Rightarrow y_1^2(1-x^2) = m^2y^2$

Diff. both sides w.r.t. $x$

$$y_1^2(-2x) + (1-x^2)(2y_1y_2) = 2m^2 y y_1$$

$$\Rightarrow -xy_1 + (1-x^2)y_2 = m^2 y$$

$$\Rightarrow \boxed{(1-x^2)y_2 - xy_1 = m^2 y}$$

Ex $y = e^{m\sin^{-1}x}$ prove that $(1-x^2)y_2 - xy_1 = m^2 y$

(7) $y = (\sin^{-1}x)^2$ prove that $(1-x^2)y_2 - xy_1 - 2 = 0$

Sol$^n$:→ $y = (\sin^{-1}x)^2$

$$\Rightarrow y_1 = 2\sin^{-1}x\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow y_1\sqrt{1-x^2} = 2\sin^{-1}x$$

$$\Rightarrow y_1^2(1-x^2) = 4(\sin^{-1}x)^2$$

$$\Rightarrow y_1^2(1-x^2) = 4y$$

$$\Rightarrow (1-x^2)(2y_1y_2) - 2xy_1^2 = 4y_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = 2$$

$$\Rightarrow \boxed{(1-x^2)y_2 - xy_1 - 2 = 0}$$

# Partial Differentiation

Suppose $Z = f(x, y)$

$\frac{\partial Z}{\partial x}$ is called partial derivative of Z w.r.t. x

$\frac{\partial Z}{\partial y}$ " " " " " " y.

$$\frac{\partial Z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial Z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$\frac{\partial Z}{\partial x}$ means Differentiation of Z w.r.t. x Keeping other variables constant.

$\boxed{\text{e.g}}$ $Z = x^2 + y^2$

$$\frac{\partial Z}{\partial x} = 2x, \quad \frac{\partial Z}{\partial y} = 2y.$$

$$\frac{\partial^2 Z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial x}\right) \qquad \frac{\partial^2 Z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y}\right)$$

$$\frac{\partial^2 Z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial x}\right) \qquad \frac{\partial^2 Z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial y}\right).$$

(2) $z = \tan^{-1}\left(\frac{x}{y}\right)$.

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= \frac{y^2}{y^2+x^2}\cdot\frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$= \frac{-x}{x^2+y^2}.$$

(3) $z = \ln(xy)$

$$\frac{\partial z}{\partial x} = \frac{1}{xy}(y) = \frac{1}{x}.$$

$$\frac{\partial z}{\partial y} = \frac{1}{xy}(x) = \frac{1}{y}$$

(4) $z = e^{ax^2+2hxy+by^2}$

$$\frac{\partial z}{\partial x} = e^{ax^2+2hxy+by^2}\left(2ax+2hy\right).$$

$$\frac{\partial z}{\partial y} = e^{ax^2+2hxy+by^2}\left(2by+2hx\right).$$

(5) $Z = \sin(x^2+y^2)$.

$$\frac{\partial Z}{\partial x} = \cos(x^2+y^2)\,(2x).$$

$$\frac{\partial Z}{\partial y} = \cos(x^2+y^2)\,(2y).$$

(6) $Z = \ln(x^2+y^2)$

$$\frac{\partial Z}{\partial x} = \frac{1}{x^2+y^2}(2x)$$

$$\frac{\partial Z}{\partial y} = \frac{2y}{x^2+y^2}$$

(7) $Z = a^{\frac{x}{y}}$

$$\frac{\partial Z}{\partial x} = a^{\frac{x}{y}}(\ln a)\cdot\frac{1}{y}$$

$$\frac{\partial Z}{\partial y} = a^{\frac{x}{y}}(\ln a)\left(\frac{-x}{y^2}\right).$$

(8) $z = x^2+2xy+y^2$

$$\frac{\partial Z}{\partial x} = 2x+2y \qquad \frac{\partial^2 Z}{\partial x^2} = 2$$

$$\frac{\partial Z}{\partial y} = 2y+2x \qquad \frac{\partial^2 Z}{\partial y^2} = 2$$

$$\frac{\partial^2 Z}{\partial y\,\partial x} = 2 \qquad \frac{\partial^2 Z}{\partial x\,\partial y} = 2.$$

(9) $u = e^{x^2+y^2+z^2}$

$$\frac{\partial u}{\partial x} = e^{x^2+y^2+z^2}(2x)$$

$$\frac{\partial u}{\partial y} = e^{x^2+y^2+z^2}(2y).$$

$$\frac{\partial u}{\partial z} = e^{x^2+y^2+z^2}(2z).$$

(10) $U = e^{xyz}$

$$\frac{\partial u}{\partial x} = e^{xyz}(yz).$$

$$\frac{\partial v}{\partial y} = e^{xyz}(zx)$$

$$\frac{\partial u}{\partial z} = e^{xyz}(xy).$$

Integration :→ The process of finding the function when it's derivative is given is called integration/ ~~Anti derivative~~. The function found is known as integral of the given function.

Simple Integration formulae

suppose $\frac{d}{dx}(g(x)) = f(x)$.

$\Rightarrow \int f(x)\,dx = g(x) + c$.

Since $\frac{d}{dx}(g(x)+c) = \frac{d}{dx}(g(x))$.

① $\int x^n\,dx = \frac{x^{n+1}}{n+1} + K \quad (n \neq -1)$

② $\int \frac{dx}{x} = \ln|x| + K$

③ $\int \cos x\,dx = \sin x + K$.

④ $\int \sin x\,dx = -\cos x + K$

⑤ $\int \sec^2 x\,dx = \tan x + K$

⑥ $\int \operatorname{cosec}^2 x\,dx = -\cot x + K$

$$\int \sec x \tan x \, dx = \sec x + K$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + K.$$

$$\int e^x \, dx = e^x + K$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + K.$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + K \text{ or } -\cos^{-1} x + K$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + K \text{ or } -\cot^{-1} x + K$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + K \text{ or } -\operatorname{cosec}^{-1} x + K.$$

## Algebra of Integrals

① $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

② $\int \lambda f(x) \, dx = \lambda \int f(x) \, dx$, $\lambda$ is a constant.

$\boxed{\text{e.g}}$ ① $\int (x^6 + x^2 + x + 1) \, dx = \int x^6 dx + \int x^2 dx + \int x \, dx + \int 1 \, dx$

$$= \frac{x^7}{7} + \frac{x^3}{3} + \frac{x^2}{2} + x + K.$$

② $\int \left(\sqrt{y} + \frac{1}{\sqrt{y}} + \frac{1}{y^2} + \frac{1}{y^3}\right) dy = \int \sqrt{y} \, dy + \int \frac{dy}{\sqrt{y}} + \int \frac{dy}{y^2}$

$$+ \int \frac{dy}{y^3}$$

$$= \int y^{1/2} dy + \int y^{-1/2} dy + \int y^{-2} dy + \int y^{-3} dy$$

$$= \frac{2}{3} y^{3/2} + 2y^{1/2} + (-1) y^{-1} + (-2) y^{-2} + C$$

(2)

(3) $\int\left(4\cos x - 3e^x + \frac{2}{\sqrt{1-x^2}}\right)dx$

$= 4\int \cos x\,dx - 3\int e^x dx + 2\int \frac{dx}{\sqrt{1-x^2}}$

$= 4(\sin x) - 3e^x + 2\sin^{-1}x + C.$

(4) $\int 6x^3(x+5)^2\,dx$

$= \int 6x^3(x^2+10x+25)\,dx$

$= \int 6x^5 dx + \int 60x^4 dx + \int 150x^3 dx$

$= x^6 + 12x^5 + \frac{150x^4}{4} + K. = x^6 + 12x^5 + \frac{75x^4}{2} + K.$

(5) $\int 5\tan^2 x\,dx$

$= \int 5(\sec^2 x - 1)\,dx$

$= 5\left[\int \sec^2 x\,dx - \int dx\right]$

$= 5\left[\tan x - x\right] + K.$

(6) $\int \frac{x^4 dx}{x^2+1} = \int\left(x^2 - 1 + \frac{1}{x^2+1}\right)dx$

$= \int x^2 dx - \int dx + \int \frac{dx}{x^2+1}$

$= \frac{x^3}{3} - x + \tan^{-1}x + K.$

(7) $\int \frac{dx}{1-\cos^2 x}$

$= \int \frac{dx}{\sin^2 x}$

$= \int \operatorname{cosec}^2 x \, dx$

$= -\cot x + K$

(8) $\int \frac{\sin x}{\cos^2 x} dx$

$= \int \tan x \sec x \, dx$

$= \sec x + C.$

(9) $\int \left( \frac{e^{2x}+1}{e^x} \right) dx$

$= \int (e^x + e^{-x}) \, dx$

$= \int e^x dx + \int e^{-x} dx$

$= e^x - e^{-x} + C.$

(10) $\int \sec^2 x \operatorname{cosec}^2 x \, dx$

$= \int \frac{dx}{\sin^2 x \cos^2 x}$

$= \int \left( \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \right) dx$

$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx$

$= \tan x - \cot x + K.$

(11) $\int \frac{\cos 3x \cos 2x + \sin 3x \sin 2x}{1-\cos^2 x} dx$

$= \int \frac{\cos(3x-2x)}{\sin^2 x} dx$

$= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} dx$

$= \int \operatorname{cosec} x \cot x \, dx$

$= -\operatorname{cosec} x + C$.

(12) Find the unique antiderivative $F(x)$ of $f(x) = 2x^2+1$ where $F(0) = -2$.

Solⁿ:→ $F(x) = \int (2x^2+1) dx$

$= 2\int x^2 dx + \int 1 dx + K$

$= \frac{2x^3}{3} + x + K$.

$F(0) = K = -2$

$\therefore F(x) = \frac{2}{3}x^3 + x - 2$

$\boxed{Ex}$ (1) $\int \left( \frac{x^4+x^3+x^2+x+2}{x^2+1} \right) dx$ (2) $\int \left( \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}} \right) dx$

(3) $\int \frac{x^6 dx}{x^2+1}$

# Integration by Substitution

$\boxed{\text{Form-I}}$ $\int f(g(x))\, g'(x)\, dx$

put $g(x) = t \Rightarrow g'(x)\,dx = dt$

$\therefore \int f(t)\, dt = \int f(g(x))\, g'(x)\, dx$

$\boxed{\text{Form-II}}$ $\int \{f(x)\}^n f'(x)\, dx$

put $f(x) = t \Rightarrow f'(x)\,dx = dt$

$\rightarrow \int t^n\, dt = \frac{t^{n+1}}{n+1} + c = \frac{\{f(x)\}^{n+1}}{n+1} + c$

$\boxed{\text{Form-III}}$ $\int \frac{f'(x)}{f(x)}\, dx$

put $f(x) = t \Rightarrow f'(x)\,dx = dt$

$\rightarrow \int \frac{dt}{t} = \ln|t| + C = \ln|f(x)| + c$

$\boxed{\text{eg}}$. $\int (ax+b)^n\, dx, \quad n \neq -1$

put $ax + b = t$

$\Rightarrow a = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{a}$

$= \frac{1}{a}\int t^n\, dt = \frac{1}{a}\,\frac{t^{n+1}}{n+1} + C$

$= \frac{(ax+b)^{n+1}}{a(n+1)} + C, \quad n \neq -1$

(2) $\int \cos(ax+b)dx = \frac{\sin(ax+b)}{a} + c$

(3) $\int \sin(ax+b)dx = \frac{-\cos(ax+b)}{a} + c$

(4) $\int \sec^2(ax+b)dx = \frac{\tan(ax+b)}{a} + c$

(5) $\int \operatorname{cosec}^2(ax+b)dx = \frac{-\cot(ax+b)}{a} + c$

(6) $\int \sec(ax+b)\tan(ax+b)dx = \frac{\sec(ax+b)}{a} + c.$

(7) $\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = \frac{-\operatorname{cosec}(ax+b)}{a} + c.$

(8) $\int e^{ax+b}dx = \frac{e^{ax+b}}{a} + c.$

(9) $\int \frac{dx}{\sqrt{1-(ax+b)^2}} = \frac{\sin^{-1}(ax+b)}{a} + c.$

(10) $\int \frac{dx}{1+(ax+b)^2} = \frac{\tan^{-1}(ax+b)}{a} + c.$

ex

(1) $\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{a\,dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c.$

(2) $\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{d(\sin x)}{\sin x} = \ln|\sin x| + c.$

(3) $\int \tan x\,dx = \int \frac{\sec x \tan x\,dx}{\sec x} = \int \frac{d(\sec x)}{\sec x} = \ln|\sec x| + c.$

(4) $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$

put $\sec x + \tan x = t$

$\Rightarrow \sec x \tan x + \sec^2 x = \frac{dt}{dx}$

$\Rightarrow \sec x(\sec x + \tan x)\, dx = dt$

$= \int \frac{dt}{t} = \ln|t| + c = \ln|\sec x + \tan x| + c$

ex prove that $\int \sec x \, dx = \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(5) $\int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)\, dx}{\operatorname{cosec} x - \cot x}$

put $\operatorname{cosec} x - \cot x = t$

$\Rightarrow -\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x = \frac{dt}{dx}$

$\Rightarrow \operatorname{cosec} x(\operatorname{cosec} x - \cot x)\, dx = dt$

$= \int \frac{dt}{t} = \ln|t| + c$

$= \ln|\operatorname{cosec} x - \cot x| + c$

ex prove that $\int \operatorname{cosec} x \, dx = \ln\left|\tan\frac{x}{2}\right| + c$

Q:→ $\int x^2 \cos(x^3)\,dx$

put $x^3 = t$

$\Rightarrow 3x^2 = \frac{dt}{dx} \Rightarrow x^2 dx = \frac{dt}{3}$

$= \frac{1}{3}\int \cos(t)\,dt$

$= \frac{1}{3}\sin(t) + C = \frac{\sin(x^3)}{3} + C.$

Q:→ $\int e^x \sec(e^x)\tan(e^x)\,dx$

put $e^x = t \Rightarrow e^x dx = dt$

$= \int \sec(t)\tan(t)\,dt$

$= \sec(t) + C$

$= \sec(e^x) + C.$

Q:→ $\int \frac{\operatorname{cosec}^2(\ln x)}{x}\,dx$

put $\ln x = t \Rightarrow \frac{dx}{x} = dt$

$= \int \operatorname{cosec}^2(t)\,dt$

$= -\cot(t) + C$

$= -\cot(\ln x) + C.$

Q:→ $\int e^{x^3} x^2 \, dx$

put $x^3 = t$

$\Rightarrow 3x^2 = \frac{dt}{dx}$

$\Rightarrow x^2 dx = \frac{dt}{3}$

$= \frac{1}{3}\int e^t \, dt$

$= \frac{1}{3} e^t + c$

$= \frac{e^{x^3}}{3} + c$

Q:→ ⓐ $\int e^{2x+7} dx = \frac{e^{2x+7}}{2} + c$

ⓑ $\int e^{x/3} dx = 3e^{x/3} + c$

ⓒ $\int e^{2\tan x} (\sec^2 x) dx$

put $\tan x = t$

$\Rightarrow \sec^2 x = \frac{dt}{dx}$

$\Rightarrow \sec^2 x \, dx = dt$

$= \int e^{2t} dt$

$= \frac{e^{2t}}{2} + c$

$= \frac{e^{2(\tan x)}}{2} + c$

Q:→ $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$ (Imp)

Sol:→ $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$

put $e^x-e^{-x}=t$

$\Rightarrow e^x+e^{-x}=\frac{dt}{dx}$

$\Rightarrow (e^x+e^{-x})dx=dt$

$= \int \frac{dt}{t} = \ln|t|+c = \ln|e^x-e^{-x}|+c$

Q:→ $\int \frac{dx}{x[1+(\ln x)^2]}$

Sol:→ $\int \frac{dx}{x[1+(\ln x)^2]}$

put $\ln x = t$

$\Rightarrow \frac{1}{x}=\frac{dt}{dx}$

$\Rightarrow \frac{dx}{x}=dt$

$= \int \frac{dt}{1+t^2}$

$= \tan^{-1}(t)+c$

$= \tan^{-1}(\ln x)+c.$

# Integration of ~~set~~ trigonometric functions

$$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right].$$

$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right].$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right].$$

Q:→ (1) $\int \sin 3x \cos 2x \, dx = \frac{1}{2}\int \left[\sin(5x) + \sin x\right] dx$

$$= \frac{1}{2}\left[-\frac{\cos(5x)}{5} - \cos(x)\right] + C.$$

(2) $\int \cos 3x \cos 4x \, dx = \frac{1}{2}\int \left[\cos 7x + \cos x\right] dx$

$$= \frac{1}{2}\left[\frac{\sin(7x)}{7} + \sin x\right] + K.$$

Higher powers of sine and cosine may be simplified to sum of sines and cosines using multiple angle formula.

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$$

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x).$$

$$\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{8}(3 + 4\cos 2x + \cos 4x).$$

Q:→ $\int \sin^2 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{2}\left[\int dx - \int \cos 2x\, dx\right]$

$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + c.$

Q:→ $\int \cos^3 x\, dx = \frac{1}{4}\int (3\cos x + \cos 3x)\, dx$

$= \frac{1}{4}\left[3\sin x + \frac{\sin(3x)}{3}\right] + c.$

Q:→ $\int \sin^4 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2$

$= \frac{1}{4}\left[\int \{1 - 2\cos 2x + \cos^2(2x)\}\, dx\right]$

$= \frac{1}{4}\left[\int dx - 2\int \cos 2x\, dx + \int \frac{1+\cos(4x)}{2}\, dx\right].$

$= \frac{1}{4}\left[x - 2\frac{\sin(2x)}{2} + \frac{1}{2}\left\{x + \frac{\sin(4x)}{4}\right\}\right] + k.$

Q:→ $\int \sin^3 x \cos^3 x\, dx$

put $\sin x = t \Rightarrow \cos x\, dx = dt$

$= \int t^3 \cos^2 x (\cos x)\, dx$

$= \int t^3 (1-t^2)\, dt = \int (t^3 - t^5)\, dt$

$= \frac{t^4}{4} - \frac{t^6}{6} + c$

$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + c.$

Q:-> $\int \cos^5 x \, dx$

$= \int \cos^4 x \cdot \cos x \, dx$

put $\sin x = t$

$\Rightarrow \cos x \, dx = dt$

$= \int (1-t^2)^2 \, dt$

$= \int (1-2t^2+t^4) dt$

$= \int dt - 2\int t^2 dt + \int t^4 dt$

$= t - 2\frac{t^3}{3} + \frac{t^5}{5} + K$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + K.$

Q:-> $\int \sin^6 x \, dx$

$= \int \left(\frac{1-\cos 2x}{2}\right)^3 dx.$

$= \frac{1}{8}\int \left[1 - \cos^3(2x) - 3\cos 2x + 3\cos^2(2x)\right] dx$

$= \frac{1}{8}\left[\int dx - \int \cos^3(2x) dx - 3\int \cos 2x \, dx + 3\int \cos^2(2x) \, dx\right].$

$= \frac{1}{8}\left[x - \int \left(\frac{3\cos(2x) - \cos(6x)}{4}\right) dx - 3\frac{\sin(2x)}{2} + 3\int \left(\frac{1+\cos 4x}{2}\right) dx\right]$

$= \frac{1}{8}\left[x - \frac{3}{8}\sin(2x) + \frac{\sin(6x)}{24} + \frac{3}{2 \cdot 2}\cos(2x) + \frac{3x}{2} + \frac{3}{8}\sin(4x) + K.\right.$

# Integration by trigonometric substitution ⑧

① $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+K$.

② $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+K$.

③ $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right|+K$.

④ $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|+K$.

Q:→ ① $\int \frac{dx}{\sqrt{25-16x^2}} = \int \frac{dx}{\sqrt{5^2-(4x)^2}} = \frac{1}{4}\sin^{-1}\left(\frac{4x}{5}\right)+c$.

② $\int \frac{e^x\,dx}{e^{2x}+9}$ put $e^x = t \Rightarrow e^x\,dx = dt$

$$= \int \frac{dt}{t^2+9} = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)+c = \frac{1}{3}\tan^{-1}\left(\frac{e^x}{3}\right)+c$$

③ $\int \frac{\cos\theta\,d\theta}{\sqrt{4\sin^2\theta+1}} = \int \frac{dt}{\sqrt{1+4t^2}}$ if we put $\sin\theta = t$

$$\cancel{\tan^{-1}(2t)+c}$$

$$\cancel{= \frac{1}{2}\tan^{-1}(2\sin\theta)+c}$$

$$= \frac{1}{2}\ln\left|t+\sqrt{1+4t^2}\right|+c$$

$$= \frac{1}{2}\ln\left|\sin\theta+\sqrt{1+4\sin^2\theta}\right|+C.$$

(4)

$$\int \frac{dx}{x\sqrt{x^8-4}}$$

$$= \frac{1}{4}\int \frac{4x^3\,dx}{x^4\sqrt{x^8-4}}$$

put $x^4 = t \Rightarrow 4x^3\,dx = dt$

$$= \frac{1}{4}\int \frac{dt}{t\sqrt{t^2-2^2}}$$

$$= \frac{1}{4}\cdot\frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right) + K$$

$$= \frac{1}{8}\sec^{-1}\left(\frac{x^4}{2}\right) + K.$$

)

$$\int \frac{(x+5)\,dx}{\sqrt{x^2+6x-7}}$$

$$= \int \frac{x+3+2}{\sqrt{(x+3)^2-16}}\,dx$$

$$= \int \frac{z+2}{\sqrt{z^2-16}} \quad \text{put } x+3 = z$$

$$= \int \frac{z\,dz}{\sqrt{z^2-16}} + 2\int \frac{dz}{\sqrt{z^2-16}}$$

Rest left as exercise.

Q:→ $\int \frac{x+3\, dx}{\sqrt{9-x^2}}$

$= \int \frac{x\, dx}{\sqrt{9-x^2}} + 3\int \frac{dx}{\sqrt{3^2-x^2}}$

$= \frac{-1}{2} \int \frac{-2x\, dx}{\sqrt{9-x^2}} + 3 \sin^{-1}\left(\frac{x}{3}\right) + K$

put $9-x^2 = v$

$\Rightarrow -2x\, dx = dv$

$= \frac{-1}{2} \int \frac{dv}{\sqrt{v}} + 3 \sin^{-1}\left(\frac{x}{3}\right) + K$

$= -\sqrt{v} + 3 \sin^{-1}\left(\frac{x}{3}\right) + K$

$= -\sqrt{9-x^2} + 3 \sin^{-1}\left(\frac{x}{3}\right) + K$

Q:→ $\int \frac{(x+3)\, dx}{\sqrt{5-x^2-4x}}$

$= \int \frac{(x+3)\, dx}{\sqrt{5+4-4-x^2-4x}}$

$= \int \frac{(x+3)\, dx}{\sqrt{9-(x^2+4x+4)}}$

$= \int \frac{(x+3)\, dx}{\sqrt{9-(x+2)^2}}$

$= \int \frac{(x+2)+1}{\sqrt{9-(x+2)^2}} dx$

$$= \int \frac{(x+2)\,dx}{\sqrt{3^2-(x+2)^2}} + \int \frac{dx}{\sqrt{3^2-(x+2)^2}}$$

put $3^2-(x+2)^2 = v$

$\Rightarrow -2(x+2) = \frac{dv}{dx}$

$\Rightarrow (x+2)dx = \frac{-dv}{2}$

$$= \frac{-1}{2}\int \frac{dv}{\sqrt{v}} + \sin^{-1}\left(\frac{x+2}{3}\right) + K$$

$$= -\int \frac{dv}{2\sqrt{v}} + \sin^{-1}\left(\frac{x+2}{3}\right) + K$$

$$= -\sqrt{v} + \sin^{-1}\left(\frac{x+2}{3}\right) + K$$

$$= -\sqrt{9-(x+2)^2} + \sin^{-1}\left(\frac{x+2}{3}\right) + K.$$

$:\to \int \frac{x^9}{x^{20}+4}\,dx$

$$= \int \frac{x^9\,dx}{(x^{10})^2+2^2}$$

put $x^{10} = t$

$\Rightarrow 10x^9 = \frac{dt}{dx}$

$\Rightarrow x^9 dx = \frac{dt}{10}$

$$= \frac{1}{10}\int \frac{dt}{t^2+2^2}$$

$$= \frac{1}{10}\cdot\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + K$$

$$= \frac{1}{20}\tan^{-1}\left(\frac{x^{10}}{2}\right) + K$$

(4)

Q:→ $\int \frac{(x+5)\,dx}{x^2+6x+13}$

$= \int \frac{(x+5)\,dx}{x^2+6x+9+4}$

$= \int \frac{(x+5)\,dx}{(x+3)^2+2^2}$

$= \int \frac{(x+3)+2}{(x+3)^2+2^2}\,dx$

$= \int \frac{(x+3)\,dx}{(x+3)^2+2^2} + 2\int \frac{dx}{(x+3)^2+2^2}$

$= \frac{1}{2}\int \frac{2(x+3)\,dx}{(x+3)^2+2^2} + \frac{2}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + K$

$= \frac{1}{2}\int \frac{dv}{v} + \tan^{-1}\left(\frac{x+3}{2}\right) + K$

where $v = (x+3)^2+2^2 \Rightarrow 2(x+3)dx = dv$

$= \frac{1}{2}\ln|v| + \tan^{-1}\left(\frac{x+3}{2}\right) + K$

$= \frac{1}{2}\ln\left|x^2+6x+13\right| + \tan^{-1}\left(\frac{x+3}{2}\right) + K$

Q:→ $\int \frac{dx}{\sqrt{3x^2+4}} = \int \frac{dx}{\sqrt{(\sqrt{3}x)^2+2^2}} = \frac{1}{\sqrt{3}}\ln\left|\sqrt{3}\,x+\sqrt{3x^2+4}\right| + K$

Q:→ $\int \frac{4e^x dx}{\sqrt{3e^{2x}+4}}$

$= 4\int \frac{e^x dx}{\sqrt{(\sqrt{3}e^x)^2+2^2}}$

put $\sqrt{3}e^x = t$

$\Rightarrow \sqrt{3}e^x dx = dt$

$= \frac{4}{\sqrt{3}} \int \frac{dt}{\sqrt{t^2+4}}$

$= \frac{4}{\sqrt{3}} \log|t+\sqrt{t^2+4}| + K$

$= \frac{4}{\sqrt{3}} \log|\sqrt{3}e^x + \sqrt{3e^{2x}+4}| + K.$

Q:→ $\int \frac{x^2 dx}{\sqrt{x^6+a^6}}$

$= \frac{1}{3}\int \frac{3x^2 dx}{\sqrt{(x^3)^2+(a^3)^2}}$

put $x^3 = v$

$\Rightarrow 3x^2 dx = dv$

$= \frac{1}{3}\int \frac{dv}{\sqrt{v^2+(a^3)^2}}$

$= \frac{1}{3} \log|v+\sqrt{v^2+a^6}| + K$

$= \frac{1}{3} \log|x^3+\sqrt{x^6+a^6}| + K.$

Q:→ $\int \frac{2x+11 \; dx}{\sqrt{x^2+10x+29}}$

$= \int \frac{2x+11 \; dx}{\sqrt{x^2+10x+25+4}}$

$= \int \frac{2x+11 \; dx}{\sqrt{(x+5)^2+2^2}}$

$= \int \frac{\{2(x+5)+1\} dx}{\sqrt{(x+5)^2+2^2}}$

$= \int \frac{2(x+5)\, dx}{\sqrt{(x+5)^2+2^2}} + \int \frac{dx}{\sqrt{(x+5)^2+2^2}}$

put $(x+5)^2+2^2 = v$

$\Rightarrow 2(x+5)dx = dv$

$= \frac{2}{2}\int \frac{dv}{2\sqrt{v}} + \ln\left|(x+5)+\sqrt{(x+5)^2+2^2}\right| + K$

$= \frac{2}{2} 2\sqrt{v} + \ln\left|(x+5)+\sqrt{(x+5)^2+2^2}\right| + K$

$= 2\sqrt{x^2+10x+29} + \ln\left|(x+5)+\sqrt{x^2+10x+29}\right| + K.$

$\Rightarrow \int \frac{e^{5x}dx}{\sqrt{e^{10x}-4}}$

put $e^{5x}=t$

$\Rightarrow 5e^{5x}=\frac{dt}{dx}$

$\Rightarrow e^{5x}dx=\frac{dt}{5}$

$= \frac{1}{5}\int \frac{dt}{\sqrt{t^2-4}}$

$= \frac{1}{5}\ln\left|t+\sqrt{t^2-4}\right|+K$

$= \frac{1}{5}\ln\left|e^{5x}+\sqrt{e^{10x}-4}\right|+K$

$\Rightarrow \int \frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{\csc^2\theta-4}} = \int \frac{\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}d\theta}{\sqrt{\csc^2\theta-4}}$

$= \int \frac{\csc\theta\cot\theta\, d\theta}{\sqrt{\csc^2\theta-4}}$

put $\csc\theta=t$

$\Rightarrow -\csc\theta\cot\theta=\frac{dt}{d\theta}$

$\Rightarrow \csc\theta\cot\theta\, d\theta=-dt$

$= -\int\frac{dt}{\sqrt{t^2-4}}$

$= -\log\left|t+\sqrt{t^2-4}\right|+K$

~~$-\log|\csc\theta+\sqrt{\csc^2\theta}$~~

$= -\log\left|\csc\theta+\sqrt{\csc^2\theta-4}\right|+C$

$\cancel{= \int \frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{\frac{1}{\sin^2\theta}-4}}}$

$\cancel{= \int \frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{\frac{1-4\sin^2\theta}{\sin\theta}}}}$

$\cancel{= \int \frac{\cos\theta\, d\theta}{\sin\theta\sqrt{1-(2\sin\theta)^2}}}$

~~put $2\sin\theta=t$~~

~~$\Rightarrow 2\cos\theta=\frac{dt}{d\theta} \Rightarrow \cos\theta\, d\theta=\frac{dt}{2}$~~

$\cancel{= \int\frac{dt}{(2t)\sqrt{1-(2t)^2}}}$

put $2t=v$

$\Rightarrow dt=\frac{dv}{2}$

$= \frac{1}{\;}\int \frac{dv}{\;}$

Integration by parts

If $v$ and $w$ are differentiable functions of $x$ then $\frac{d}{dx}(vw) = v\frac{dw}{dx} + w\frac{dv}{dx}$

$$\Rightarrow v\frac{dw}{dx} = \frac{d}{dx}(vw) - w\frac{dv}{dx}$$

$$\Rightarrow \int v\frac{dw}{dx}dx = \int\frac{d}{dx}(vw)dx - \int w\frac{dv}{dx}\cdot dx$$

Setting $u = \frac{dw}{dx} \Rightarrow w = \int u\,dx$

$$\Rightarrow \int uv\,dx = vw - \int\left[\int u\,dx\cdot\frac{dv}{dx}\right]dx$$

$$\Rightarrow \int uv\,dx = v\int u\,dx - \int\left[\int u\,dx\cdot\frac{dv}{dx}\right]dx.$$

$$\int(\text{Int of product})dx = (\text{1st function})\int(\text{2nd function})\,dx - \int\left[\frac{d}{dx}(\text{1st function})\left(\int \text{2nd function}\right)\right]dx$$

So 1st function and 2nd function is not that significant.

Take a ~~lineword~~ I LATE

I → Inverse trigonometric functions
L → Logarithm functions
A → Algebraic function
T → Trigonometric function
E → Exponential function

Whichever letter comes first take that as first and other as second function.

e.g. $\int (\log x)\,dx = \int (\log x)\cdot 1\,dx$

$\log x$ is logarithmic function

1 is algebraic function

So $\log x$ is 1st and 1 is 2nd function

$$= (\log x)\int 1\,dx - \int \left[\frac{d}{dx}(\log x)\left(\int 1\,dx\right)\right]dx$$

$$= (\log x)(x) - \int \left(\frac{1}{\cancel{x}}\right)(\cancel{x})\,dx$$

$$= x\log x - \int dx + K$$

$$= \boxed{x\log x - x + K}$$

Q:→ $\int x \cos x \, dx$

$= (x) \int \cos x \, dx - \int \left[\frac{d}{dx}(x) \int \cos x \, dx\right] dx$

$= (x)(\sin x) - \int (1)(\sin x) \, dx$

$= x(\sin x) - \int \sin x \, dx$

$= x(\sin x) - (-\cos x) + K$

$= x(\sin x) + \cos x + K.$

Q:→ $\int x e^x \, dx$

$= (x) \int e^x \, dx - \int \left[\frac{d}{dx}(x) \int e^x \, dx\right] dx$

$= (x)(e^x) - \int (1) \cdot e^x \, dx$

$= x e^x - \int e^x \, dx + K$

$= x e^x - e^x + K$ — ①

:→ $\int x^2 e^x \, dx$

$= (x^2) \int e^x \, dx - \int \left[\frac{d}{dx}(x^2) \int e^x \, dx\right] dx$

$= (x^2) \cdot e^x - \int (2x) \cdot e^x \, dx$

$= x^2 e^x - 2 \int x e^x \, dx$

$= x^2 e^x - 2\left[x e^x - e^x\right] + K$

$= x^2 e^x - 2x e^x + 2e^x + K.$

Ex Verify $\int x^n e^x dx$

$= (x^n)\cdot e^x - n(x^{n-1})\cdot e^x + {}^nC_2\, x^{n-2} e^x + \cdots + n!\, e^x + K.$

Q:→ $\int \tan^{-1}x \, dx$

$= \int (\tan^{-1}x)\cdot 1\, dx$

~~(1) f~~ $= \tan^{-1}(x)\int 1\,dx - \int\left[\frac{d}{dx}(\tan^{-1}x)\int 1\,dx\right]dx$

$= \tan^{-1}(x)\cdot x - \int\left(\frac{1}{1+x^2}\right)(x)\,dx$

$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x\,dx}{1+x^2}$

$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + K.$

Imp $\int e^x[f(x)+f'(x)]\,dx = e^x f(x) + c.$

Look $\frac{d}{dx}[e^x f(x)] = e^x f'(x) + f(x)e^x$

$= e^x(f(x)+f'(x))$

$\therefore \boxed{\int e^x[f(x)+f'(x)]\,dx = e^x f(x) + K}$

Q:→ $\int \frac{e^x}{x}(1+x\ln x)\,dx$

$= \int e^x\left(\frac{1}{x}+\ln x\right)dx$

Look $f(x)=\ln x \Rightarrow f'(x)=\frac{1}{x}$

$= \int e^x\left(\ln x + \frac{d}{dx}(\ln x)\right)dx$

$= e^x \ln x + K.$

Q:→ $\int e^x(\tan x + \ln \sec x)\,dx$

easy take as exercise Ans:→ $e^x \ln(\sec x) + K.$

Q:→ $\int e^x(\cot x + \ln \sin x)\,dx = e^x \ln(\sin x) + K.$

Q:→ $\int \frac{x e^x\,dx}{(1+x)^2}$

$= \int \frac{(1+x-1)e^x\,dx}{(1+x)^2}$

$= \int e^x\left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\}dx$

put $\frac{1}{1+x} = f(x) \Rightarrow \frac{-1}{(1+x)^2} = f'(x)$

$$\therefore \int e^x\left[\frac{1}{1+x}-\frac{1}{(1+x)^2}\right]dx$$

$$= \frac{e^x}{1+x}+K.$$

Q:→ $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

$$= \int e^x\left(\frac{1}{1+\cos x}+\frac{\sin x}{1+\cos x}\right)dx$$

Let $f(x)=\frac{\sin x}{1+\cos x}$

$$f'(x)=\frac{(1+\cos x)\cos x-\sin x(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$$

$$=\frac{1+\cos x}{(1+\cos x)^2}=\frac{1}{1+\cos x}$$

$$\therefore \int e^x\left(\frac{1}{1+\cos x}+\frac{\sin x}{1+\cos x}\right)dx$$

$$=\frac{e^x\sin x}{1+\cos x}+K.$$

Q:-> $\int e^{ax} \cos bx \, dx = I$ (say)

$$= \cos(bx) \cdot \int e^{ax} dx - \int \left[ \frac{d}{dx}(\cos bx) \cdot \int e^{ax} dx \right] dx$$

$$= \cos(bx) \cdot \frac{e^{ax}}{a} - \int (-b \sin(bx)) \frac{e^{ax}}{a} dx$$

$$= \frac{e^{ax} \cos(bx)}{a} + \frac{b}{a} \int e^{ax} \sin(bx) dx.$$

$$= \frac{e^{ax} \cos(bx)}{a} + \frac{b}{a} \left[ \sin(bx) \int e^{ax} dx - \int \left[ \frac{d}{dx}(\sin bx) \int e^{ax} dx \right] dx \right]$$

$$= \frac{e^{ax} \cos(bx)}{a} + \frac{b}{a} \left[ \frac{\sin(bx) \cdot e^{ax}}{a} - \int b \cos(bx) \cdot \frac{e^{ax}}{a} dx \right].$$

$$= \frac{e^{ax} \cos(bx)}{a} + \frac{b}{a^2} e^{ax} \sin(bx) - \frac{b^2}{a^2} I$$

$$\Rightarrow \left( I + \frac{b^2}{a^2} I \right) = e^{ax} \left[ \frac{b \sin bx + a \cos bx}{a^2} \right].$$

$$\Rightarrow \left( \frac{a^2+b^2}{\cancel{a^2}} \right) I = \frac{e^{ax} (a \cos bx + b \sin bx)}{\cancel{a^2}}$$

$$\Rightarrow \boxed{I = e^{ax} \left( \frac{a \cos bx + b \sin bx}{a^2 + b^2} \right) + K}$$

$\boxed{\text{Ex}}$ Find $\int e^{ax} \sin(bx)\, dx = ?$

Ans:$\rightarrow$ $\dfrac{e^{ax}}{a^2+b^2}\left(a \sin bx - b\cos bx\right) + K.$

$\boxed{\text{Formula}}$

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + K.$$

$$\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + K$$

$$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + K$$

$\boxed{\text{Proof}}$

$$I = \int \sqrt{a^2-x^2}\, dx = \sqrt{a^2-x^2}\int 1\,dx - \int\left[\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)\int 1\,dx\right]dx$$

$$= \sqrt{a^2-x^2}\,(x) - \int \frac{(-\cancel{2}x)}{\cancel{2}\sqrt{a^2-x^2}}\cdot x\, dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{-x^2\, dx}{\sqrt{a^2-x^2}}$$

$$= x\sqrt{a^2-x^2} - \int \frac{(a^2-x^2-a^2)\,dx}{\sqrt{a^2-x^2}}$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\, dx + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$I + a^2 \sin^{-1}\left(\frac{x}{a}\right).$$

$$= x\sqrt{a^2-x^2} - I + a^2\sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore\ 2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right).$$

$$\Rightarrow \boxed{I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)}$$

Other two parts are easy and similar to the last one. work out them.

---

Q:→ $\int \ln(x+\sqrt{x^2+a^2})\,dx$

$= \ln(x+\sqrt{x^2+a^2})\int 1\,dx - \int\left[\frac{d}{dx}\ln(x+\sqrt{x^2+a^2})\int 1\,dx\right]dx$

$= x\ln(x+\sqrt{x^2+a^2}) - \int\left[\frac{1}{x+\sqrt{x^2+a^2}}\left(1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}}\right)\cdot x\right]dx$

$= x\ln(x+\sqrt{x^2+a^2}) - \int\left(\frac{1}{\cancel{x+\sqrt{x^2+a^2}}}\right)\left(\frac{\cancel{x+\sqrt{x^2+a^2}}}{\sqrt{x^2+a^2}}\right)x\,dx$

$= x\ln(x+\sqrt{x^2+a^2}) - \frac{1}{2}\int\frac{2x\,dx}{\sqrt{x^2+a^2}}$

$= x\ln(x+\sqrt{x^2+a^2}) - \int\frac{dv}{2\sqrt{v}}$ put $v = x^2+a^2$ $\Rightarrow 2x\,dx = dv$

$= x\ln(x+\sqrt{x^2+a^2}) - \sqrt{v} + C$

$= x\ln(x+\sqrt{x^2+a^2}) - \sqrt{x^2+a^2} + C$

---

$\boxed{ex}$ $\int \ln(x+\sqrt{x^2-a^2})\,dx.$

Q: → ① $\int \sqrt{9-x^2}\,dx$

② $\int \sqrt{5-4x^2}\,dx$

Both are easy exercises. Use formula to get them.

③ $\int \sqrt{1-x^2-2x}\,dx$

$= \int \sqrt{1-x^2-2x-1+1}\,dx$

$= \int \sqrt{2-(x^2+2x+1)}\,dx$

$= \int \sqrt{(\sqrt{2})^2-(x+1)^2}\,dx$

put $(x+1) = t \Rightarrow dx = dt$

$= \int \sqrt{(\sqrt{2})^2-t^2}\,dt$

Rest is easy and work out.

④ $\int e^{2z} \sqrt{e^{4z}+6}\,dz$

$= \int e^{2z} \sqrt{(e^{2z})^2+(\sqrt{6})^2}\,dz$

put $e^{2z} = v \Rightarrow 2e^{2z}\,dz = dv$

$= \frac{1}{2}\int \sqrt{v^2+(\sqrt{6})^2}\,dv$

Use formula to get answer.

(5) $\int \sec^2\theta \sqrt{\sec^2\theta + 3}\, d\theta$

$= \int \sec^2\theta \sqrt{1 + \tan^2\theta + 3}\, d\theta$

$= \int \sec^2\theta \sqrt{\tan^2\theta + 4}\, d\theta$

put $\tan\theta = v$

$\Rightarrow \sec^2\theta\, d\theta = dv$

$= \int \sqrt{v^2 + 2^2}\, dv$

Rest is easy.

(6) $\int \sqrt{x^2 - 4x + 2}\, dx$

$= \int \sqrt{x^2 - 4x + 4 - 2}\, dx$

$= \int \sqrt{(x-2)^2 - (\sqrt{2})^2}\, dx$

put $x - 2 = t \Rightarrow dx = dt$

$= \int \sqrt{t^2 - (\sqrt{2})^2}\, dt$

Rest is easy.

(7) $\int e^{3x} \cos 2x \, dx$ (8) $\int e^{2x} \sin x \, dx$

Both you should work out. We have already solved general questions $\int e^{ax} \cos(bx) dx$ and $\int e^{ax} \sin(bx) dx$.

(9) $\int (2x^2+1) e^{x^2} dx$

put $x^2 = t$

$\Rightarrow 2x\,dx = dt$

$\Rightarrow dx = \dfrac{dt}{2\sqrt{t}}$

$= \int (2t+1) e^{t} \dfrac{dt}{2\sqrt{t}}$

$= \int e^{t} \left( \sqrt{t} + \dfrac{1}{2\sqrt{t}} \right) dt$

lwk $\dfrac{d}{dt}(\sqrt{t}) = \dfrac{1}{2\sqrt{t}}$

$\therefore \int e^{t} \left( \sqrt{t} + \dfrac{1}{2\sqrt{t}} \right) dt = e^{t} \sqrt{t} + K$

$= e^{x^2} (x) + K.$

(10) $\int \left[ \dfrac{1}{\ln x} - \dfrac{1}{(\ln x)^2} \right] dx$ put $\ln x = t \Rightarrow x = e^{t}$

$\Rightarrow dx = e^{t} dt$

$\int \left( \dfrac{1}{t} - \dfrac{1}{t^2} \right) e^{t} dt = \dfrac{e^{t}}{t} + C = \dfrac{x}{\ln x} + C.$

Q→ $\int \sin(\ln x)\,dx$

put $\ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$= \int \sin(t)\cdot e^t\,dt$

Now use standard by parts technique.

Q:→ $\int \sin x \ln(\operatorname{cosec}x - \cot x)\,dx$.

$= \ln(\operatorname{cosec}x - \cot x)\int \sin x\,dx - \int\left[\frac{d}{dx}\left(\ln(\operatorname{cosec}x - \cot x)\right)\int \sin x\,dx\right]dx$

$= \ln(\operatorname{cosec}x - \cot x)(-\cos x) - \int\left(\frac{\operatorname{cosec}^2x - \operatorname{cosec}x\cot x}{\operatorname{cosec}x - \cot x}\right)(-\cos x)\,dx$

$= -\cos x\ln(\operatorname{cosec}x - \cot x) + \int(\cos x)\left(\frac{\operatorname{cosec}x\,\cancel{(\operatorname{cosec}x - \cot x)}}{\cancel{(\operatorname{cosec}x - \cot x)}}\right)dx$

$= -\cos x\ln(\operatorname{cosec}x - \cot x) + \int\frac{\cos x}{\sin x}\,dx$

$= -\cos x\ln(\operatorname{cosec}x - \cot x) + \ln\sin x + K$

## Definite Integral

### Fundamental theorem of Integral Calculus

If $f(x)$ is continuous on $[a,b]$ and $F(x) = \int f(x)\,dx$

then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof:-> $\int_a^b f(x)\,dx = \left[F(x)+K\right]_a^b$

$= (F(b)+K) - (F(a)+K)$

$= F(b) - F(a)$.

—x—

· $\int_a^b \{g(x) \pm h(x)\}\,dx = \int_a^b g(x)\,dx \pm \int_a^b h(x)\,dx$

· $\int_a^b \lambda g(x)\,dx = \lambda \int_a^b g(x)\,dx$.

$\boxed{\text{e.g}}$

(1) $\int_1^2 x^3\,dx = \left(\frac{x^4}{4}\right)_1^2 = \frac{1}{4}(2^4 - 1^4) = \frac{15}{4}$.

(2) $\int_0^{\pi/2} \sin x\,dx = -(\cos x)_0^{\pi/2} = -(\cos \pi/2 - \cos 0)$

$= -(0-1) = 1$.

(3) $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(x)\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4} - 0 = \pi/4$.

(4) $\int_0^{\pi/2} x\cos x\,dx = \left[x\int \cos x\,dx - \int\left\{\frac{d}{dx}(x)\int \cos x\,dx\right\}dx\right]_0^{\pi/2}$

$= \left[x(\sin x) - \int (1)\cdot \sin x\,dx\right]_0^{\pi/2}$

$= \left\{x(\sin x) - (-\cos x)\right\}_0^{\pi/2}$

$$= \left[\left\{\frac{\pi}{2}\sin(\pi/2) + \cos(\pi/2)\right\} - \left\{0 + \cos(0)\right\}\right]$$

$$= \left(\frac{\pi}{2} + 0\right) - (0 + 1)$$

$$= \frac{\pi}{2} - 1$$

(5) $\int_0^{\pi/2} (3x^2 + 2x + \cos x)\, dx \rightarrow$ Easy left as exercise.

(6) $\int_2^3 2x\, e^{x^2} dx$

put $x^2 = t$

$\Rightarrow 2x\, dx = dt$

$$\int_4^9 e^t\, dt$$

$$= \left(e^t\right)_4^9 = e^9 - e^4.$$

(7) $\int_0^{\pi/4} \sin^5 x \cos x\, dx = \int_0^{\frac{1}{\sqrt{2}}} z^5\, dz \quad (\sin x = z)$

$$= \left(\frac{z^6}{6}\right)_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{6}\left[\left(\frac{1}{\sqrt{2}}\right)^6\right]$$

$$= \frac{1}{6}\left[\frac{1}{8}\right] = \frac{1}{48}.$$

## Elementary properties of definite Integral

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

**Proof** :→ Let $F'(x) = f(x)$

$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

$\int_b^a f(x)\,dx = [F(x)]_b^a = F(a) - F(b)$

$\therefore \boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$

(ii) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz$

Proof is easy. i.e. definite Integral is independent of symbol of variable of integration

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad a < c < b$

**Proof** :→ $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = |F(x)|_a^c + |F(x)|_c^b$

$= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$

$= |F(x)|_a^b$

$= \int_a^b f(x)\,dx.$

g

① $\int_1^4 [x]\,dx$

$= \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + \int_3^4 [x]\,dx$

$= \int_1^2 dx + \int_2^3 2\,dx + \int_3^4 3\,dx$

$= (2-1) + 2(3-2) + 3(4-3)$

$= 1+2+3 = 6.$

② $\int_{-3}^4 |x|\,dx$

$= \int_{-3}^0 |x|\,dx + \int_0^4 |x|\,dx$

$= -\int_{-3}^0 x\,dx + \int_0^4 x\,dx$

$= -\left(\frac{x^2}{2}\right)_{-3}^0 + \left(\frac{x^2}{2}\right)_0^4$

$= \frac{-1}{2}(0-9) + \frac{1}{2}(16-0)$

$= \frac{9}{2} + \frac{16}{2} = \frac{25}{2}.$

③ $\int_0^1 (2x+1)^4\,dx$

$= \left.\frac{(2x+1)^5}{10}\right|_0^1$

$= \frac{1}{10}\left[3^5 - 1^5\right] = \frac{242}{10} = \frac{121}{5}.$

(5) $\int_0^1 x^7 (1+x^8)^{1/3} \, dx$.

put $1+x^8 = t$ $\quad t = 1$ when $x=0$

$t = 2$ when $x = 1$

$\Rightarrow 8x^7 = \frac{dt}{dx}$

$\Rightarrow x^7 dx = \frac{dt}{8}$

$\frac{1}{8}\int_1^2 dt \, t^{1/3}$

$= \frac{1}{8}\left[\frac{t^{4/3}}{4/3}\right]_1^2$

$= \frac{1}{8}\left[\frac{3}{4} t^{4/3}\right]_1^2$

$= \frac{3}{32}\left(2^{4/3} - 1\right)$.

(6) $\int_0^{\pi/2} (\cos x - \sin x) \, dx$

$= \int_0^{\pi/2} \cos x \, dx - \int_0^{\pi/2} \sin x \, dx$

$= [\sin x]_0^{\pi/2} - [-\cos x]_0^{\pi/2}$

$= [\sin(\pi/2) - \sin 0] + [\cos \pi/2 - \cos 0]$

$= (1-0) + (0-1) = 0$.

(7) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{1+\cos 2x}{1-\cos 2x}\,dx$

$$= \int_{\pi/6}^{\pi/3} \frac{\cancel{2}\cos^2 x}{\cancel{2}\sin^2 x}\,dx$$

$$= \int_{\pi/6}^{\pi/3} \cot^2 x\,dx$$

$$= -\operatorname{cosec}(x)\Big|_{\pi/6}^{\pi/3}$$

$$= -\left[\operatorname{cosec}(\pi/3) - \operatorname{cosec}(\pi/6)\right]$$

$$= -\left[\frac{2}{\sqrt{3}} - 2\right]$$

$$= 2 - \frac{2}{\sqrt{3}}$$

(8) $\displaystyle\int_{0}^{3/2} [2x]\,dx$

$$= \int_0^{1/2} [2x]\,dx + \int_{\frac{1}{2}}^{1} [2x]\,dx + \int_1^{3/2} [2x]\,dx$$

$0 \le x < \frac{1}{2} \Rightarrow 0 \le 2x < 1 \Rightarrow [2x] = 0$

$\frac{1}{2} \le x < 1 \Rightarrow 1 \le 2x < 2 \Rightarrow [2x] = 1$

$1 \le x < 3/2 \Rightarrow 2 \le 2x < 3 \Rightarrow [2x] = 2$

$$= 0\int_0^{1/2} dx + \int_{\frac{1}{2}}^{1} dx + 2\int_1^{3/2} dx$$

$$= \left(1-\frac{1}{2}\right) + 2\left(\frac{3}{2}-1\right) = \frac{1}{2} + 2\left(\frac{1}{2}\right) = 1+\frac{1}{2}$$

$$= \frac{3}{2}$$

# Some More properties of definite Integral

① $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

In particular $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

② $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(-x) = f(x) \\ 0 & \text{if } f(-x) = -f(x) \end{cases}$$

③ $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

Q:→ $\int_1^2 \frac{\sqrt{x}\,dx}{\sqrt{3-x}+\sqrt{x}}$

Let $I = \int_1^2 \frac{\sqrt{x}\,dx}{\sqrt{3-x}+\sqrt{x}} = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}$ using ①

$\therefore 2I = \int_1^2 \left(\frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\right) dx$

$= \int_1^2 dx = (x)_1^2 = 2-1 = 1$

$\therefore \boxed{I = \frac{1}{2}}$

Q:→ $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$

put $x = a\sin\theta$ $\quad x=0 \Rightarrow \theta=0,\ x=a \Rightarrow \theta=\pi/2$

$\Rightarrow dx = a\cos\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{a\cos\theta\, d\theta}{a\sin\theta + \sqrt{a^2-a^2\sin^2\theta}}$$

$$= \int_0^{\pi/2} \frac{a\cos\theta\, d\theta}{a\sin\theta + \sqrt{a^2\cos^2\theta}}$$

$$= \int_0^{\pi/2} \frac{a\cos\theta\, d\theta}{a\sin\theta + a\cos\theta}$$

$$= \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sin\theta + \cos\theta} = \int_0^{\pi/2} \frac{\cos(\pi/2-\theta)\, d\theta}{\sin(\pi/2-\theta)+\cos(\pi/2-\theta)}$$

$$= \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\cos\theta + \sin\theta}$$

$$\therefore 2I = \int_0^{\pi/2} \left(\frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} d\theta = (\theta)_0^{\pi/2} = \pi/2$$

$$\therefore \boxed{I = \pi/4}$$

Q:→ $\int_0^{\pi/2} \frac{\sin\theta \, d\theta}{\sin\theta + \cos\theta} = \int_0^{\pi/2} \frac{\cos\theta \, d\theta}{\sin\theta + \cos\theta}$ (Imp)

$= \int_0^{\pi/2} \frac{d\theta}{1+\cot\theta} = \int_0^{\pi/2} \frac{d\theta}{1+\tan\theta} = \pi/4$.

All exercises are same and proof is available in the last exercise solved.

Q:→ $\int_0^{\pi/2} \frac{\sqrt{\sin\theta}\, d\theta}{\sqrt{\sin\theta}+\sqrt{\cos\theta}} = \int_0^{\pi/2} \frac{\sqrt{\cos\theta}\, d\theta}{\sqrt{\cos\theta}+\sqrt{\sin\theta}} = \int_0^{\pi/2} \frac{d\theta}{1+\sqrt{\tan\theta}} = \int_0^{\pi/2} \frac{d\theta}{1+\sqrt{\cot\theta}}$

Use property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Ans:→ $\pi/4$

Q:→ $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$ (Imp)

$I = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta = \int_0^{\pi/4} \ln[1+\tan(\pi/4-\theta)]\, d\theta$

$= \int_0^{\pi/4} \ln\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right] d\theta = \int_0^{\pi/4} \ln\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \{\ln(2) - \ln(1+\tan\theta)\}\, d\theta$

$= \ln(2)\int_0^{\pi/4} d\theta - \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$

$= \pi/4 \ln 2 - I \Rightarrow 2I = \pi/4 \ln 2$

$\Rightarrow I = \frac{\pi}{8}\ln 2$.

Q:→ $\int_0^{\pi/2} \frac{\sin\theta \, d\theta}{\sin\theta + \cos\theta} = \int_0^{\pi/2} \frac{\cos\theta \, d\theta}{\sin\theta + \cos\theta}$ (Imp)

$= \int_0^{\pi/2} \frac{d\theta}{1+\cot\theta} = \int_0^{\pi/2} \frac{d\theta}{1+\tan\theta} = \pi/4$

All exercises are same and proofs available in the last exercise solved.

Q:→ $\int_0^{\pi/2} \frac{\sqrt{\sin\theta}\, d\theta}{\sqrt{\sin\theta}+\sqrt{\cos\theta}} = \int_0^{\pi/2} \frac{\sqrt{\cos\theta}\, d\theta}{\sqrt{\cos\theta}+\sqrt{\sin\theta}} = \int_0^{\pi/2} \frac{d\theta}{1+\sqrt{\tan\theta}} = \int_0^{\pi/2} \frac{d\theta}{1+\sqrt{\cot\theta}}$

Use property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

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$I = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta = \int_0^{\pi/4} \ln[1+\tan(\pi/4-\theta)]\, d\theta$

$= \int_0^{\pi/4} \ln\left[1 + \frac{1-\tan\theta}{1+\tan\theta}\right] d\theta = \int_0^{\pi/4} \ln\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \{\ln(2) - \ln(1+\tan\theta)\}\, d\theta$

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$= \pi/4 \ln 2 - I \Rightarrow 2I = \pi/4 \ln 2$

$\Rightarrow I = \frac{\pi}{8} \ln 2.$

Q:→ $\int_0^{\pi/2} \frac{\cos x-\sin x}{1+\sin x\cos x}\,dx = 0$ (prove).

Sol:→ $f(x) = \frac{\cos x-\sin x}{1+\sin x\cos x}$

$f\left(\frac{\pi}{2}-x\right) = \frac{\cos(\pi/2-x)-\sin(\pi/2-x)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}$

$= \frac{\sin x-\cos x}{1+\sin x\cos x} = -f(x).$

$\therefore \int_0^{\pi/2} f(x)\,dx = 0 \quad \left(\because \int_0^{2a} f(x)\,dx = 0 \text{ if } f(2a-x) = -f(x)\right)$

Q:→ $\int_0^{\pi} \sin x\cos^2 x\,dx = 2\int_0^{\pi/2} \cos^2 x\sin x\,dx \quad (\because f(\pi-x)=f(x))$

$= -2\int_1^0 t^2\,dt$ put $\cos x = t$ $\Rightarrow -\sin x\,dx = dt$

$= 2\int_0^1 t^2\,dt = 2\left(\frac{t^3}{3}\right)_0^1 = \frac{2}{3}(1-0)$

$= \frac{2}{3}.$

Q:→ $\int_1^3 \frac{\sqrt{x}\,dx}{\sqrt{4-x}+\sqrt{x}}$

Sol:→ $I = \int_1^3 \frac{\sqrt{x}\,dx}{\sqrt{4-x}+\sqrt{x}} = \int_1^3 \frac{\sqrt{1+3-x}\,dx}{\sqrt{4-(1+3-x)}+\sqrt{1+3-x}}$

$= \int_1^3 \frac{\sqrt{4-x}\,dx}{\sqrt{x}+\sqrt{4-x}} = I$

$$\therefore\ 2I = \int_1^3 \left( \frac{\sqrt{x}+\sqrt{4-x}}{\sqrt{4-x}+\sqrt{x}} \right) dx$$

$$= \int_1^3 dx = (x)_1^3 = 3-1 = 2.$$

$$\Rightarrow \boxed{I = 1}.$$

Q:→ $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$.

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}\,dx}{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}\,dx}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}} = I$$

$$\therefore\ 2I = \int_{\pi/6}^{\pi/3} \left( \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} \right) dx = \int_{\pi/6}^{\pi/3} dx$$

$$= \frac{\pi}{3} - \pi/6 = \pi/6$$

$$\therefore \boxed{I = \frac{\pi}{12}}$$

Q:→ $\int_0^1 x(1-x)^{100}\,dx$

$= \int_0^1 (0+1-x)\left(1-(1-x)\right)^{100} dx$

$= \int_0^1 (1-x)(x^{100})\,dx$

$= \int_0^1 (x^{100} - x^{101})\,dx$

$= \int_0^1 x^{100}\,dx - \int_0^1 x^{101}\,dx$

$= \left.\frac{x^{101}}{101}\right]_0^1 - \left.\frac{x^{102}}{102}\right]_0^1$

$= \left(\frac{1}{101} - 0\right) - \left(\frac{1}{102} - 0\right)$

$= \frac{1}{101} - \frac{1}{102} = \frac{102-101}{(101)(102)} = \frac{1}{(101)(102)}$.

Q:→ $\int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2} \ln(\cos x)\,dx = \frac{\pi}{2}\ln\left(\frac{1}{2}\right)$.

Solⁿ:→ $I = \int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2} \ln\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx$

$= \int_0^{\pi/2} \ln(\cos x)\,dx = I$

$2I = \int_0^{\pi/2} \left\{\ln(\sin x) + \ln(\cos x)\right\} dx$

$$= \int_0^{\pi/2} \ln(\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \ln\left(\frac{2\sin x \cos x}{2}\right) dx$$

$$= \int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \ln(\sin 2x)\, dx - \int_0^{\pi/2} (\ln 2) \cdot dx$$

$$= \int_0^{\pi/2} \ln(\sin 2x)\, dx - (\ln 2) \cdot \pi/2$$

put $2x = v$

$\Rightarrow 2\,dx = dv$

$\Rightarrow dx = \frac{dv}{2}$

$$= \frac{1}{2} \int_0^{\pi} \ln(\sin v)\, dv - \pi/2 (\ln 2)$$

$$= \left(\frac{1}{\cancel{2}}\right) \cdot \cancel{2} \int_0^{\pi/2} \ln(\sin x)\, dx - \pi/2 \ln(2)$$

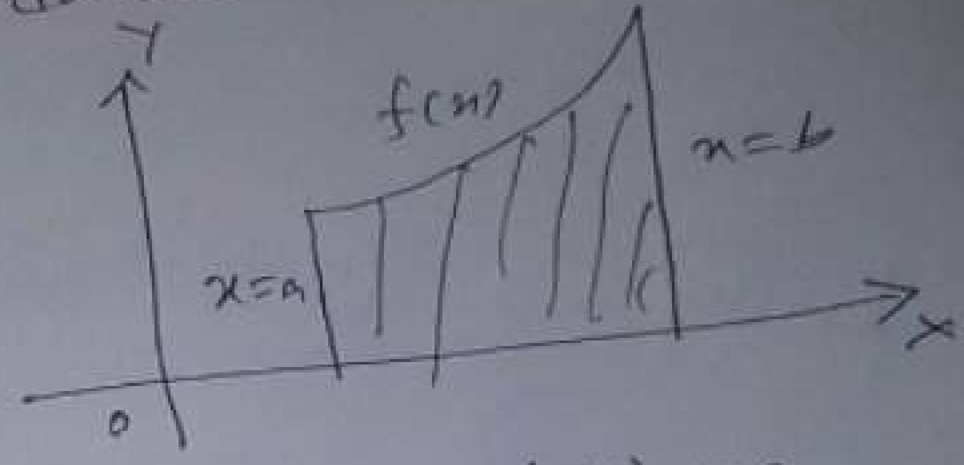
$$\Rightarrow 2I = I - \frac{\pi}{2} \ln 2$$

$$\Rightarrow I = -\frac{\pi}{2} \ln 2 \Rightarrow I = \frac{\pi}{2} \ln 2^{-1} = \frac{\pi}{2} \ln\left(\frac{1}{2}\right).$$

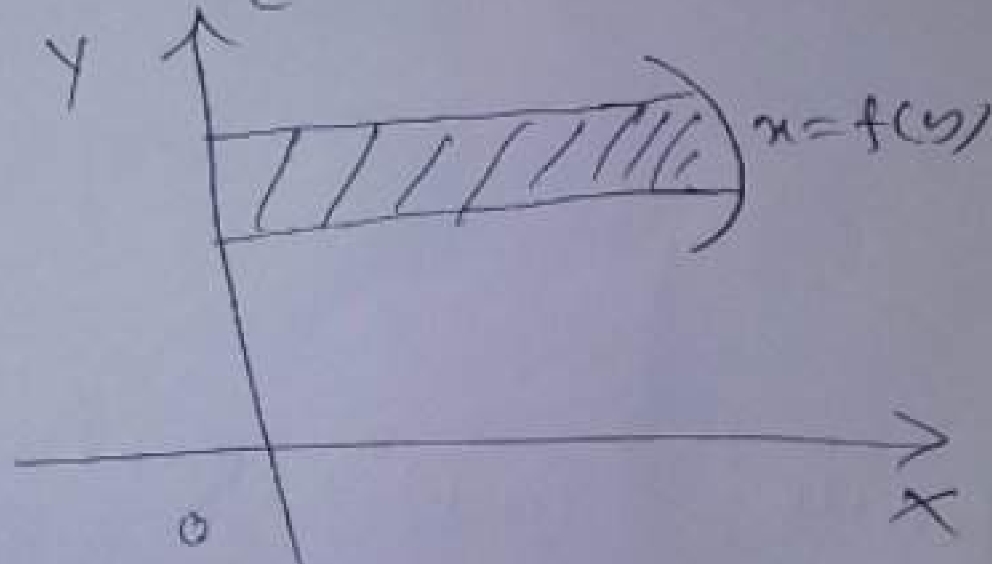
# Area under plane curves



$\int_a^b f(x)\,dx$ represents the area under $y=f(x)$ above x-axis between ordinates $x=a$ and $x=b$.




Area betn $x=f(y)$, $x=0$ and the abscissae $y=c$, $y=d$ is equal to $\int_c^d f(y)\,dy$

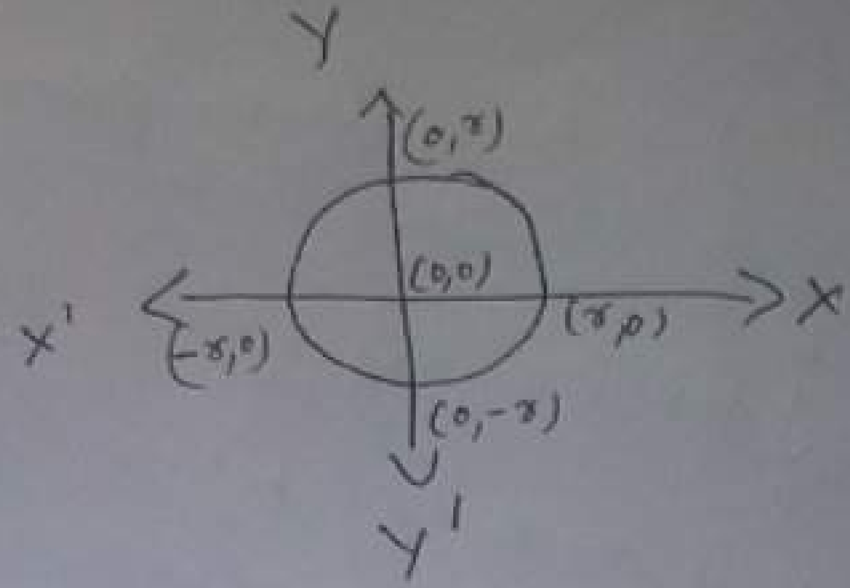



Area betn 2 curves $y=f(x)$, $y=g(x)$ with $g(x) < f(x)$ in $[a,b]$ is given by

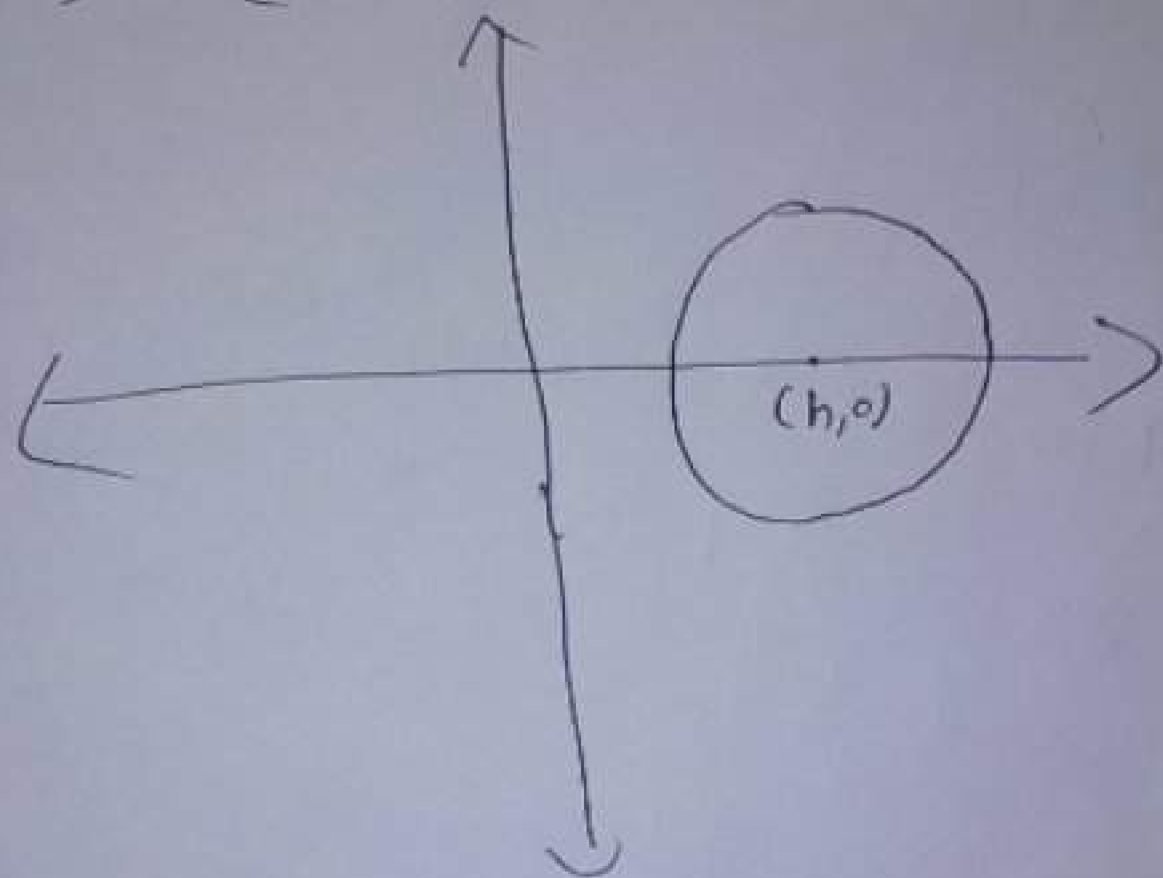
$$A = \int_a^b \{f(x) - g(x)\}\,dx.$$

Graph of standard curves

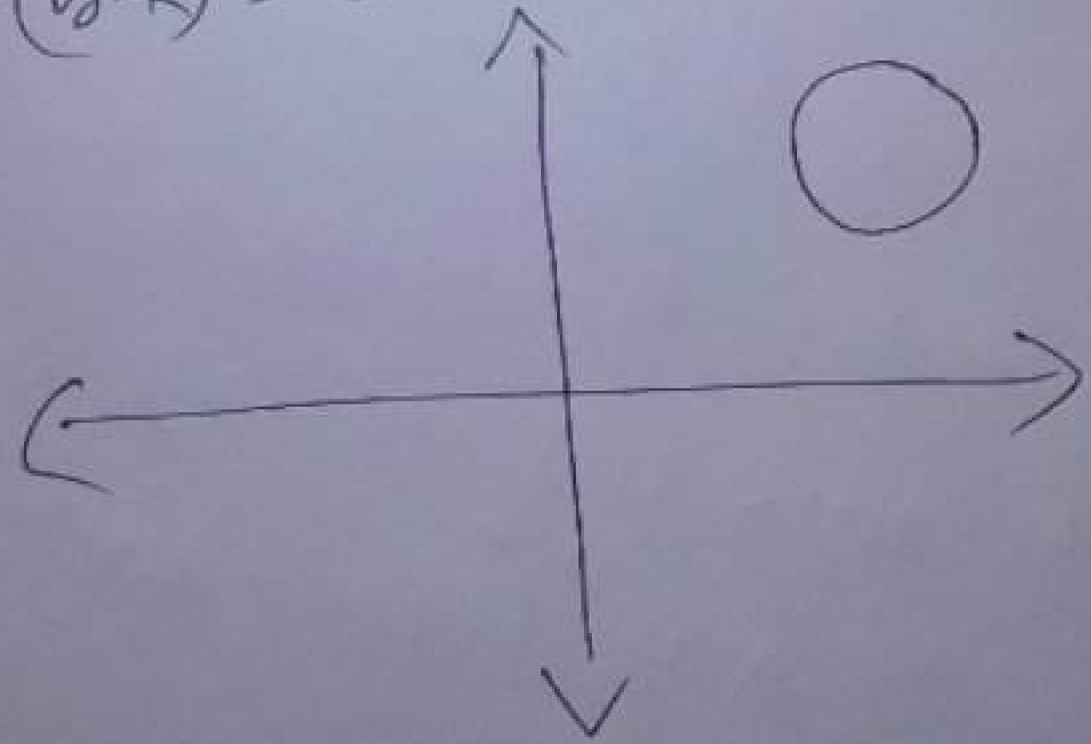
Circle: → $x^2+y^2=r^2$



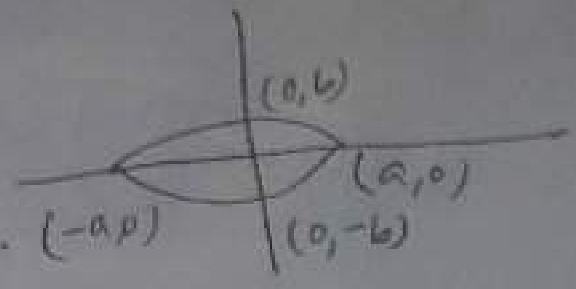

Circle → $(x-h)^2+(y-\cancel{k})^2=r^2$



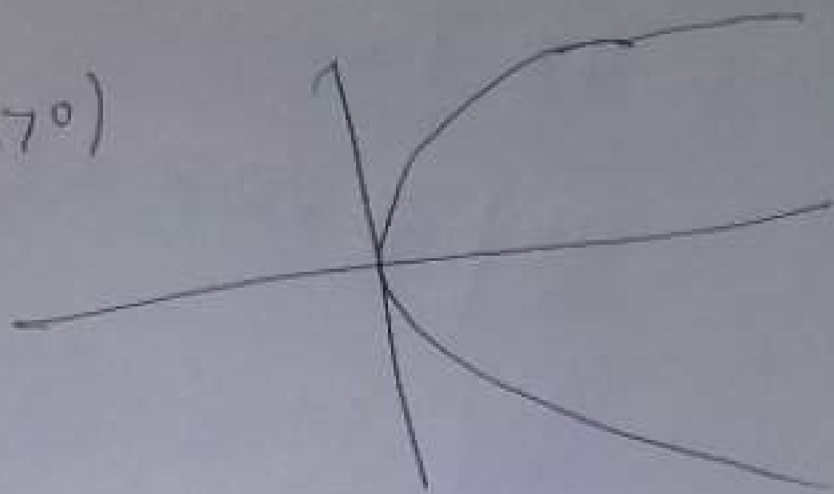

Circle $(x-h)^2+(y-k)^2=a^2$



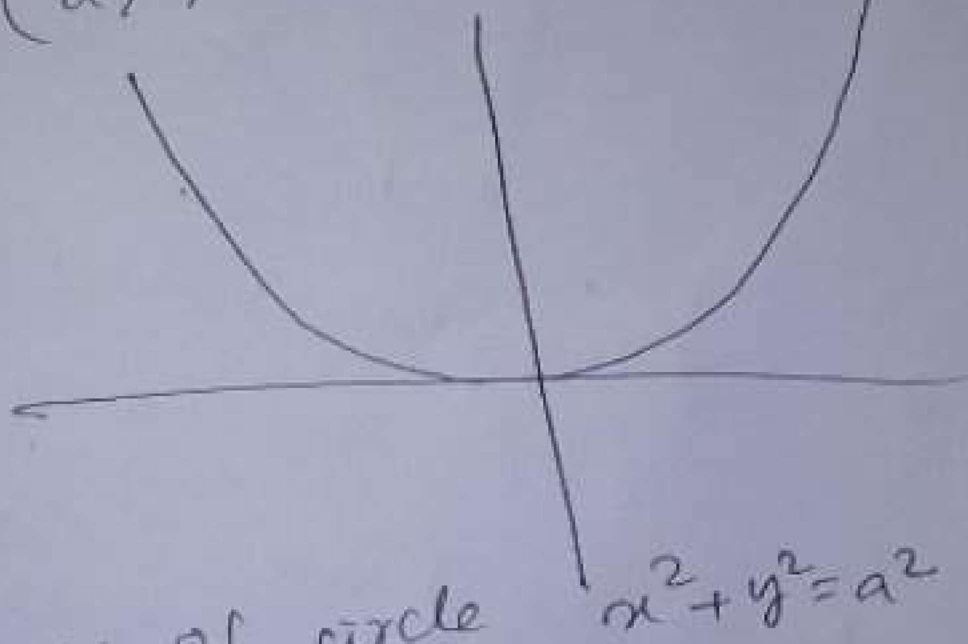
**Ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$




**parabola** $y^2 = 4ax \quad (a>0)$



$x^2 = 4ay \quad (a>0).$

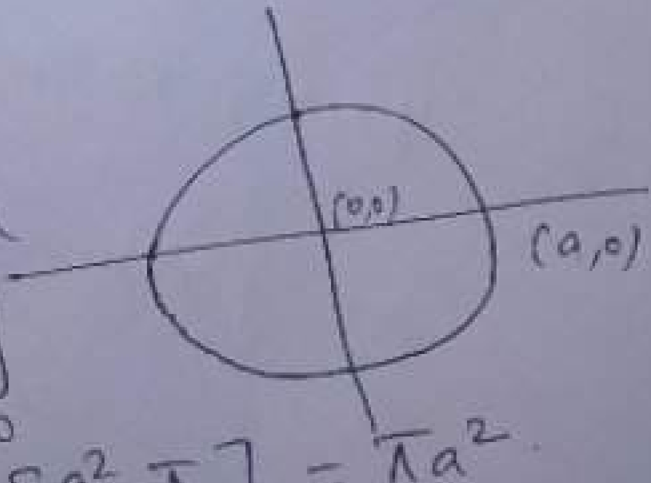


Q:→ Find the area of circle $x^2+y^2=a^2$

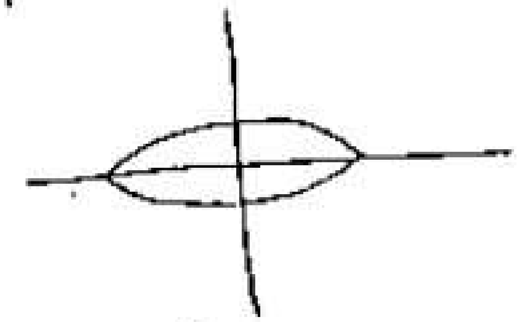
$$A = 4\int_0^a \left(\sqrt{a^2-x^2}\right) dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4\left[\frac{a^2}{2}\cdot \sin^{-1}(1)\right] = \cancel{4}\left[\frac{a^2}{\cancel{2}}\cdot\frac{\pi}{\cancel{2}}\right] = \pi a^2.$$



Q:→ Find the area of ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$$A = 4\int_0^a \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}\,dx$$

$$= 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}\left[\left\{0+\frac{a^2}{2}\sin^{-1}(1)\right\}-\left\{0+0\right\}\right]$$

$$= \frac{4b}{\cancel{a}}\cdot\frac{a^{\cancel{2}}}{\cancel{2}}\cdot\frac{\pi}{\cancel{2}}.$$

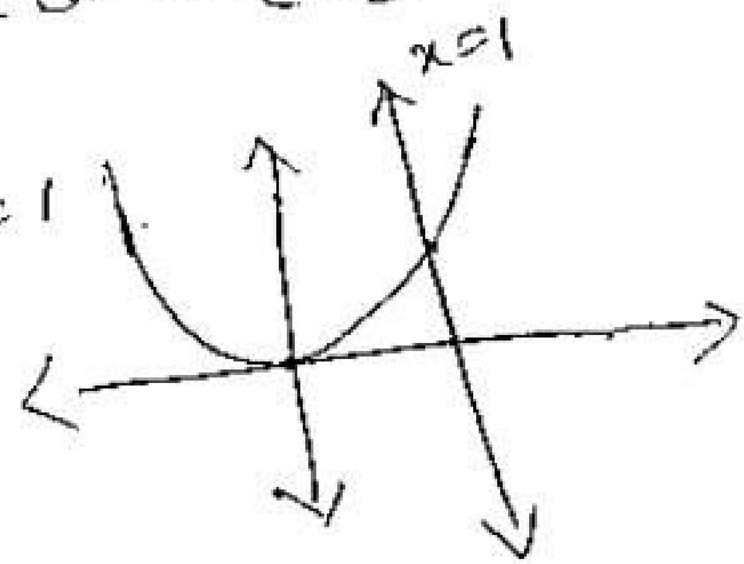
$$= \pi ab.$$

Q:→ Find area bounded by

$y=e^x,\ y=0,\ x=2,\ x=4$

Ans:→ $A=\int_2^4 y\,dx=\int_2^4 e^x\,dx=\left(e^x\right)_2^4=e^4-e^2.$

Q:→ $y=x^2,\ y=0,\ x=1$




$$A=\int_0^1 x^2\,dx=\frac{x^3}{3}\Big|_0^1$$

$$=\frac{1}{3}.$$

Q:→ $y = \sin x$, $y=0$, $x=\pi/2$.

A:→ $A = \int_0^{\pi/2} \sin x \, dx = (-\cos x)_0^{\pi/2}$

$= -(0-1) = 1.$

Q:→ $xy=a^2$, $y=0$, $x=\alpha$, $x=\beta$ $(\beta>\alpha>0)$.

A:→ Area $= \int_\alpha^\beta y\,dx = \int_\alpha^\beta \left(\frac{a^2}{x}\right) dx = a^2 \ln x \Big|_a^b$

$= a^2(\ln b - \ln a)$

$= a^2 \ln\left(\frac{b}{a}\right).$

Ex Find area enclosed by

1. $y=e^x$, $x=0$, $y=2$, $y=3$.
2. $y^2=x$, $x=0$, $y=1$
3. $xy=a^2$, $x=0$, $y=\alpha$, $y=\beta$ $(0<\alpha<\beta)$
4. $y^2=x^3$, $x=0$, $y=1$

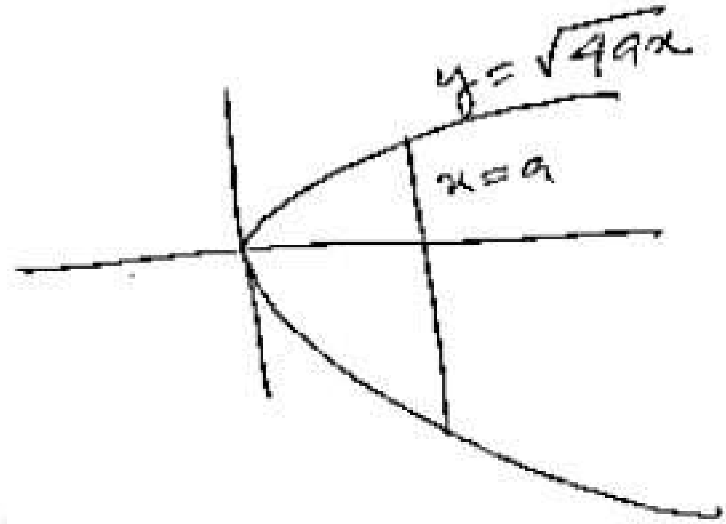
Sol$^n$ of (2) Area $= \int_0^1 x\,dy = \int_0^1 y^2 dy$

$= \frac{y^3}{3}\Big|_0^1$

$= \frac{1}{3}$

Rest You should work out.

Q:→ Find area of parabola $y^2 = 4ax$ bounded by $x = a$.




Sol$^n$:→

$$\text{Area} = 2\int_0^a y\,dx$$

$$= 2\int_0^a \sqrt{4ax}\,dx$$

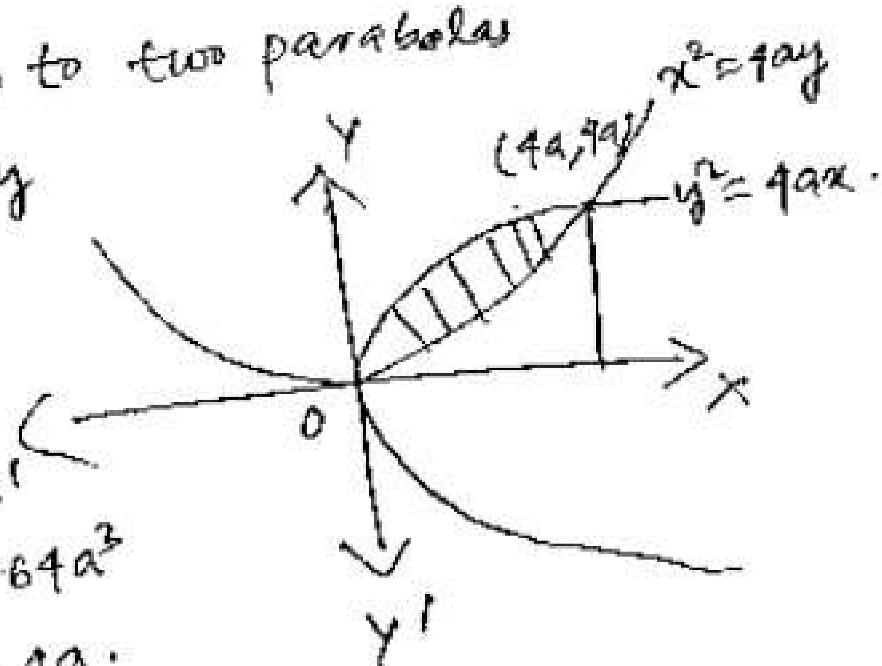
$$= 2\int_0^a (2)\sqrt{a}\,\sqrt{x}\,dx$$

$$= (4)(\sqrt{a})\left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{8}{3}(a)^{1/2}\cdot(a)^{3/2}$$

$$= \frac{8}{3}(a)^2.$$

Q:→ Find area common to two parabolas $y^2 = 4ax$ and $x^2 = 4ay$




Sol$^n$:→ ~~Area =~~ $\int$

$y^2 = 4ax \Rightarrow x = y^2/4a$

$\therefore \frac{y^4}{16a^2} = 4ay \Rightarrow y^3 = 64a^3$

$\Rightarrow y = 4a.$

$\therefore x = \frac{16a^2}{4a} = 4a$

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$$\therefore \text{Area} = \int_0^{4a} (y_1 - y_2)\, dx$$

$$= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right) dx$$

$$= \int_0^{4a} \left(2a^{1/2} x^{1/2} - \frac{1}{4a} x^2\right) dx$$

$$= 2\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right) - \frac{1}{4a} \cdot \frac{x^3}{3}\Bigg]_0^{4a}$$

$$= \left[\frac{4a^{1/2}}{3} x^{3/2} - \frac{1}{12a} \cdot x^3\right]_0^{4a}$$

$$= \frac{4a^{1/2}}{3} \cdot (4a)^{3/2} - \frac{1}{12a} (4a)^3$$

$$= \frac{4}{3} a^{1/2} (8) a^{3/2} - \frac{1}{\underset{3}{\cancel{12}}\cancel{a}} \cdot \overset{16}{\cancel{64}} a^{\overset{2}{\cancel{3}}}$$

$$= \frac{32}{3} a^2 - \frac{16}{3} a^2 = \frac{16a^2}{3}.$$

$\boxed{\text{Ex}}$ Find the area bet$^n$ parabolas $y^2 = x$ and $x^2 = y$